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Random paths to pairwise stability in many-to-many matching problems: a study on market equilibration

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Abstract This paper considers a decentralized process in many-to-many matching problems. We show that if agents on one side of the market have substitutable preferences and those on the other side have responsive preferences, then, from an arbitrary matching, there exists a finite path of matchings such that each matching on the path is formed by *satisfying* a blocking individual or a blocking pair for the previous matching, and the final matching is pairwise-stable. This implies that an associated stochastic process converges to a pairwise-stable matching in finite time with probability one, if each blocking individual or pair is satisfied with a positive probability at each period along the process.

Keywords Many-to-many matching · Pairwise stability · Stability · Random paths

JEL Classification C71 · C78

1 Introduction

Various economic interactions can be modeled as two-sided matching.[1] An example is entry-level labor markets. In such models, there are firms who are

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[1] See Roth and Sotomayor (1990) for a survey of two-sided matching models.

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seeking one or more workers to hire, and there are workers who are seeking jobs in one or more firms. Each firm has preferences over subsets of workers and each worker has preferences over subsets of firms. A matching is a solution of a two-sided matching problem. It matches firms and workers with each other. A matching is *pairwise-stable* if no firm prefers being matched to a proper subset of its current employees, no worker prefers being matched to a proper subset of her current jobs, and no firm-worker pair (who are originally not matched with each other) prefer being matched with each other, possibly instead of some of their current partners. Existing research suggests that pairwise stability is a key property in real-life markets, in the sense that centralized markets with pairwise-unstable matching mechanisms often suffer from market failures, while those with pairwise-stable mechanisms do not (Roth 1991). Many, if not most, markets, however, do not have any centralized matching mechanism, *and* they still do not seem to suffer from market failure. A reasonable conjecture is that some process of decentralized decision making will result in a stable matching. To model the decentralized process, we consider a sequence of matchings, called a *blocking path*, such that each matching on the path is formed from the previous matching after *a blocking individual or a blocking pair is satisfied* (that is, a firm or worker dissolves her partnership with her undesirable partners, or a firm and worker come together and are matched with each other, possibly instead of some of their less desirable partners). If such a blocking path converges to a stable matching, we refer to it as a *convergent path*. This paper establishes the existence of a convergent path in *many-to-many* matching problems, when agents on one side have substitutable preferences and agents on the other side have responsive preferences.

An arbitrary blocking path may not converge to a stable matching even in *one-to-one* matching problems, in which each firm and worker can have at most one partner (Knuth 1976). By contrast, Roth and Vande Vate (1990) showed the existence of a convergent path for such problems. A number of interesting results has been obtained since on the existence of convergent paths in different matching problems. In a *roommates* problem, when preferences satisfy the "no-odd rings" condition[2] and indifferences are possible, the Roth and Vande Vate path converges to a stable matching (Chung 2000). If "no-odd rings" condition is not satisfied, preferences are strict, and there exists a stable matching, then a convergent path still exists (Diamantoudi et al. 2004). On other domains, such as *many-to-one* matching problems (and so, in many-to-many matching problems), the existence of a convergent path is not guaranteed. In many-to-one matching problems with couples, when preferences are weakly responsive, a convergent path exists (Klaus and Klijn 2006).[3]

In many-to-many matching problems, agents on both sides of the market may have multiple partners. Existence or non-existence of convergent paths

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[2] "No-odd rings condition" guarantees the existence of a stable matching in a roommates problem.

[3] Also in coalition formation games, convergent paths may not exist. Pápai (2004) finds conditions that guarantee existence. Definition of stability is different from pairwise stability in this domain of problems and in many-to-one matching problems with couples.

has been unknown in such problems. Since many-to-many matching problems are non-trivial extensions of one-to-one matching problems, many properties of one-to-one matching problems do not extend to this wider class. Although a many-to-one matching problem (with responsive preferences) is isomorphic to a one-to-one matching problem (Roth and Sotomayor 1990), there is no such isomorphism for many-to-many matching problems, even under responsive preferences. For example, pairwise stability is no longer equivalent to other solution concepts introduced in the literature such as core stability, group stability or setwise stability, and pairwise-stable matchings may even be inefficient (Blair 1988; Roth and Sotomayor 1990; Sotomayor 1999, 2004; see also Echenique and Oviedo 2006 for more detailed discussions of the structure of the set of pairwise stable matchings in many-to-many matching problems).

Therefore, it is not trivial whether or not a convergent path exists for this domain of problems. Our main theorem shows the existence in many-to-many matching problems, when one side has substitutable preferences and the other side has responsive preferences. This preference restriction has a nice real-life application. When firms have responsive preferences, and workers have a class of preferences stricter than substitutable preferences (called, *categorywise-responsive preferences*), this domain represents the preferences in most of the British medical intern-hospital matching markets (Roth 1991; Konishi and Ünver 2006). As in Roth and Vande Vate (1990), the existence of a convergent path guarantees that a myopic random process of blocking converges to a pairwise-stable matching in a decentralized many-to-many matching problem, when every blocking agent and pair is satisfied with a positive probability at each stage (Remark 3).[4]

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[4] *Pairwise stability* is not equivalent to *group stability* (Definition 5.4 in Roth and Sotomayor (1990) for many-to-one matching, and Konishi and Ünver (2006) for many-to-many matching; see Jackson and van den Nouweland (2005) for usage of *group stability* in networks) or *core stability* in many-to-many matching problems, unlike in many-to-one matching problems with substitutable preferences. Then why is it still important to find a path of matchings that converges to a *pairwise-stable* matching? First, since some deviations via groups may not be credible, *group stability* may be too demanding in many-to-many matching problems. Two weaker stability concepts are considered in the literature: *setwise stability* (Roth 1984; Sotomayor 1999) and *credible group stability* (Konishi and Ünver 2006). Although *setwise stability*, *pairwise stability*, and *core stability* are not equivalent under substitutable preferences (Sotomayor 1999), when one side has responsive preferences and the other side has maximin responsive preferences, *pairwise-stable matchings* are in the *core* and *setwise stability* is equivalent to *pairwise stability* (Sotomayor 2004). When one side has *substitutable* preferences and the other side has *strongly substitutable* preferences, the equivalence between *setwise stability* and *pairwise stability* still holds (Echenique and Oviedo 2006). When one side has categorywise-responsive preferences and the other side has responsive preferences, *credible group stability* is equivalent to *pairwise stability* (Konishi and Ünver 2006). These results suggest that *pairwise stability* is a reasonable group stability concept in certain many-to-many matching problems. In Edinburgh and Cardiff regions of the British medical intern-hospital matching markets, *pairwise-stable* mechanisms have functioned smoothly for the last four decades.

1.1 The technical contribution

Our proof is substantially different from that of Roth and Vande Vate (1990) for the one-to-one matching problem, although our idea is based on theirs. They start with an arbitrary unstable matching, and let blocking agents dissolve their partnerships with their undesirable partners one at a time. Then they construct a monotonically expanding sequence of agent sets who cannot form blocking pairs among themselves (we call this property "internal stability"). Their algorithm works roughly as follows: Let F and W be sets of firms and workers, respectively. Given an internally stable set I such that agents in it are not matched with agents outside (we call this property "closure"), we add an agent outside of I, say a worker $\bar{w} \in W \setminus I$, to set I. Let $\bar{I} \equiv I \cup \{\bar{w}\}$. Now, $\bar{I}$ may not be internally stable. Then, let blocking pairs in $\bar{I}$ block the current matching one at a time repeatedly until $\bar{I}$ becomes internally stable. Typically, the new entrant $\bar{w}$ expands the set of workers available to firms in $\bar{I}$. Since (by closure of I) $\bar{w}$ does not dump any partners in $\bar{I}$, when she becomes a member of $\bar{I}$, she initiates a monotone process (at each round of which everyone in $F \cap I$ is made weakly better off, and one member of $F \cap I$ is made strictly better off). Since firms cannot be made infinitely well off, this process eventually stops. Moreover, internal stability and closure are attained for the larger set $\bar{I}$. Proceeding iteratively, the whole set $F \cup W$ becomes internally stable (and closed). That is, a pairwise-stable matching is reached in a finite number of iterations. All the results cited above use the same idea, except for Diamantoudi et al. (2004). They all construct a monotonically expanding sequence of internally stable and closed sets of agents. Such construction is difficult in many-to-many matching. Since $\bar{w}$ may have many partners, each of whom has many other partners and so on, potentially both workers *and* firms in $\bar{I}$ will be dumped, when $\bar{w}$, their partners, the partners of these partners and so on, are included in $\bar{I}$. This set expands the set of potential partners not only for firms but for workers as well, and the resulting blocking path may not be monotone.

This paper addresses the above challenge by offering two technical innovations. Suppose that firms have substitutable preferences, and workers have responsive preferences. First, we obtain a monotonically expanding sequence of internally stable sets of agents without imposing closure. Specifically, we start from an internally stable, but not necessarily closed set of agents I, and then add an *arbitrary* agent outside I, say $\bar{\imath} \in (F \cup W) \setminus I$, to obtain a larger internally stable set $\bar{I} = I \cup \{\bar{\imath}\}$.[5] Second, we develop a technique to choose the order for picking the blocking pairs on the path so that monotonicity is retained, even when a new member $\bar{\imath}$ or her partners in the blocking pairs dump agents in I. First, we treat workers in I *as if* they have not been dumped. That is, these agents block the current matching only if they would be willing to block it *even if, contrary to reality, they had never been dumped by $\bar{\imath}$ or her mates in the blocking pairs when she joined I.* This step turns out to be monotone, making every

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[5] Our technique of choosing a new member turns out to be simpler than Roth and Vande Vate's in their original domain, for they have to choose which agent to add very carefully.

worker weakly better off and some strictly better off. After every such blocking pair is satisfied, we continue by letting the remaining blocking pairs block the current matching. This is as if workers suddenly realize that they were dumped when $\bar{\imath}$ joined I. Now, this narrows the perceived sets of available partners for workers, and by responsiveness, they are now more willing to be matched with firms, but not to dump their current partners. Every firm is made weakly better off, and some firms are made strictly better off in this step, making the process monotone. Thus, we construct a process, resulting in a monotonically expanding sequence of internally stable sets. The whole set of agents becomes internally stable eventually, and we obtain a pairwise-stable matching.

One open question is whether our result extends to a more general framework, for example cases where every agent has substitutable preferences. Given that the class of substitutable preferences is a maximal domain to guarantee the existence of a stable matching (Hatfield and Milgrom 2005), this seems to be an interesting direction of research. Since for workers with substitutable preferences, our process in the last blocking stage is not monotone (that is, not every firm is necessarily made weakly better off), our result does not immediately generalize to this domain.[6]

2 The model

Let F be a set of firms and W be a disjoint set of workers. Let $N = F \cup W$ be the set of all agents. Firms are seeking one or more workers to hire, and workers are seeking employment in one or more firms. For each agent $i \in N$, let P_i denote the set of potential partners for i, i.e., $P_i = F$ if $i \in W$, and $P_i = W$ if $i \in F$. Each agent $i \in N$ has preferences over 2^{P_i}, the subsets of her set of potential partners, and her preference relation is denoted by $\succeq_i$. For each agent $i \in N$, let $\succ_i$ denote the strict preference relation derived from $\succeq_i$. We assume that for each agent $i \in N$, the relation $\succeq_i$ is a linear order, i.e., for all $S, T \subseteq P_i$, $S \succeq_i T$ if and only if $S = T$ or $S \succ_i T$. In words, each agent has strict preferences over partner sets.[7] Let $\succeq = (\succeq_i)_{i \in N}$ be the preference profile. The list $(F, W, \succeq)$ is a *many-to-many matching problem*. We fix a problem throughout the paper.

A solution of a problem is a *matching*, defined below. We use a graph representation to define a matching. Agents are nodes, and the set of nodes are fixed in a graph. There can be edges between two nodes. A *partnership* between agents $i, j \in N$ is an edge between agents i and j. It is denoted by (i,j) or (j,i).[8] A *graph* is a set of partnerships (Jackson and Wolinsky 1996). A *matching* μ is a graph such that for each $i, j \in N$, $(i,j) \in \mu \Rightarrow j \in P_i$. Let $\mathcal{M}$ be the set of matchings. For each matching $\mu \in \mathcal{M}$, and each agent $i \in N$, let $\mu(i) = \{j \in P_i \mid (i,j) \in \mu\}$

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[6] See Roth (1984); Blair (1988); Alkan (1999, 2001, 2002); Echenique and Oviedo (2006); Martínez et al. (2004) for some other properties of one-to-one problem that extend to the many-to-many problem under different preference restrictions.

[7] Strictness of preferences is assumed just for simplicity and can be relaxed.

[8] Both (i,j) and (j,i) represent the same partnership. If f is a firm and w is a worker, then we will denote a partnership between f and w by (f, w), whenever possible.

be the *partner set of i under μ*. For each agent $i \in N$, and a potential partner set $S \subseteq P_i$, the *choice of i in S* is the most preferred partner set of i among the partners in S. We denote it by $\mathrm{Ch}_i(S)$, i.e., $\mathrm{Ch}_i(S) \subseteq S$ is such that $\mathrm{Ch}_i(S) \succeq_i T$ for all $T \subseteq S$. For each matching $\mu \in \mathcal{M}$, and each agent $i \in N$, we say that *i blocks μ individually* if $\mu(i) \neq \mathrm{Ch}_i(\mu(i))$. A matching μ is *individually stable* if no agent blocks μ. For each matching $\mu \in \mathcal{M}$, and each pair $(f, w) \in (F \times W) \setminus \mu$, we say that *$(f, w)$ blocks μ in a pair* if $w \in \mathrm{Ch}_f(\mu(f) \cup w)$ and $f \in \mathrm{Ch}_w(\mu(w) \cup f)$.[9] A matching $\mu \in \mathcal{M}$ is *pairwise-stable* if μ is blocked neither individually nor in pairs.[10]

On the general strict preference domain, a pairwise-stable matching may not exist. We need restrictions on preferences to guarantee existence. For each agent $i \in N$, her preference relation $\succeq_i$ is *substitutable* if for all $S, S' \subseteq P_i$ with $S \subseteq S'$, we have $\mathrm{Ch}_i(S') \cap S \subseteq \mathrm{Ch}_i(S)$ (Kelso and Crawford 1982). That is, a partner who is chosen from a larger set of potential partners is always chosen from a smaller set of potential partners. If every agent has substitutable preferences, then there exists a pairwise-stable matching (Roth 1984).

Another class of preferences is that of responsive preferences. For each agent $i \in N$, and some positive integer q_i, her preference relation $\succeq_i$ is *responsive with quota q_i* if (i) for all $j, k \in P_i$, and all $S \subseteq P_i \setminus \{j, k\}$ with $|S| < q_i$, we have $j \cup S \succeq_i k \cup S \Leftrightarrow j \succeq_i k$, (ii) for all $j \in P_i$, and all $S \subseteq P_i \setminus j$ with $|S| < q_i$, we have $j \cup S \succeq_i S \Leftrightarrow j \succeq_i \emptyset$, and (iii) for all $S \subseteq P_i$ with $|S| > q_i$, we have $\emptyset \succ_i S$ (Roth 1985). That is, the agent's ranking of two partners (including being unmatched) is independent of her other partners, unless she exceeds her quota, and any set of partners exceeding her quota is less preferred to being unmatched. It is easy to show that any responsive preference is substitutable.

Let $\mu \in \mathcal{M}$ be pairwise-unstable. For each blocking individual $i \in N$ of μ, a matching $\mu' \in \mathcal{M}$ is *obtained from μ by satisfying i*, if

$$\mu' = \mu \setminus \{(i, j) \mid j \in \mu(i) \setminus \mathrm{Ch}_i(\mu(i))\}.$$

In the above expression, $\mu(i) \setminus \mathrm{Ch}_i(\mu(i))$ is the set of *dumped* partners by agent i, when she is satisfied.

For each blocking pair $(f, w) \in F \times W$ of μ, a matching $\mu' \in \mathcal{M}$ is *obtained from μ by satisfying (f, w)*, if

$$\begin{aligned}\mu' = {} & \mu \cup \{(f, w)\} \setminus \\ & \left(\left\{(f, w') \mid w' \in \mu(f) \setminus \mathrm{Ch}_f(\mu(f) \cup w)\right\} \cup \left\{(w, f') \mid f' \in \mu(w) \setminus \mathrm{Ch}_w(\mu(w) \cup f)\right\}\right).\end{aligned}$$

In the above expression, for all $\{i, j\} = \{f, w\}$, $\mu(i) \setminus \mathrm{Ch}_i(\mu(i) \cup j)$ is the set of partners *dumped* by agent i, when (f, w) is satisfied. We have $\mu'(i) =$

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[9] We will denote singleton $\{i\}$ as i whenever convenient.

[10] Our usage of *blocking* is different from its standard usage in coalitional form games, as in *corewise blocking*. The former refers to a situation in which an agent can unilaterally dump some existing partners, or a pair of partners can form partnership and dump some partners without the *consent and participation of their other partners*, which are required in the latter.

$\mathrm{Ch}_i\,(\mu\,(i)\cup j)$ for all $i\in\{f,w\}$. For each $I\subseteq N$, and $\mu\in\mathcal{M}$, we say that I is *internally stable under* μ if no pair or individual in I blocks μ. The following observation is important for the proof of our result:

Observation 1 *Assume that agent $i\in N$ has responsive preferences with quota q_i. Let μ be an individually stable matching, and (i,j) be a blocking pair for μ for some partner $j\in P_i$. When (i,j) is satisfied, agent i will not dump any partners if $|\mu(i)|<q_i$, and agent i will dump exactly one partner if $|\mu(i)|=q_i$. In the latter case, the partner dumped by i is the least favorite partner of i under $\mu\,(i)$.*

The following lemma will be useful in the proof of our main theorem:

Lemma 1 *Assume that agent $i\in N$ has substitutable preferences. Let μ and μ' be two matchings obtained by satisfying one blocking pair or individual at a time and such that μ' is obtained later than μ. Assume that agent i was never dumped in the process between μ and μ'. If $j\in\mathrm{Ch}_i\left(\mu'\,(i)\cup j\right)$ then $j\in\mathrm{Ch}_i\,(\mu\,(i)\cup j)$.*

Proof Since i is never dumped between μ and μ', she was made weakly better off in such a way that $\mathrm{Ch}_i\left(\mu\,(i)\cup\mu'\,(i)\right)=\mu'\,(i)$. Let $j\in P_i$ be such that $j\in\mathrm{Ch}_i\left(\mu'\,(i)\cup j\right)$. We have

$$\begin{aligned}\mathrm{Ch}_i\left(\mu'\,(i)\cup j\right)&=\mathrm{Ch}_i\left(\mathrm{Ch}_i\left(\mu\,(i)\cup\mu'\,(i)\right)\cup j\right)\\&=\mathrm{Ch}_i\left(\mu\,(i)\cup\mu'\,(i)\cup j\right).\end{aligned}$$

This, together with choice of j and substitutability of i's preferences, imply that

$$j\in\mathrm{Ch}_i\,(\mu\,(i)\cup j)\,.$$

□

3 The result

Our main result is the following:

Theorem 1 *Assume that every agent on one side of the market has substitutable preferences, and every agent on the other side has responsive preferences with quota. Let μ_0 be a pairwise-unstable matching. There exists a sequence of matchings $\{\mu_t\}_{t=0}^{T}$ such that μ_T is pairwise-stable, and for all $t\in\{0,1,2,...,T-1\}$, μ_{t+1} is obtained from μ_t by satisfying a blocking individual or a blocking pair.*

Since an individually stable matching is obtained from an individually unstable one by sequentially satisfying one individual at a time, we assume that μ_0 is *individually stable* without loss of generality. By an inductive argument, Theorem 1 results from the following lemma:

Lemma 2 *Let the assumptions be as in Theorem 1. Let $\bar{\mu}\in\mathcal{M}$ be individually stable, and $I\subsetneq N$ be internally stable under $\bar{\mu}$. There exist a set of agents*

$\bar{I}$ with $I \subsetneq \bar{I}$, and a sequence of matchings $\{\mu_s\}_{s=0}^{S}$ such that $\mu_0 = \bar{\mu}$, for all $s \in \{0, 1, 2, ..., S-1\}$, μ_{s+1} is obtained from μ_s by satisfying a blocking pair, and $\bar{I}$ is internally stable under μ_S.[11]

Proof Without loss of generality, let all firms have substitutable preferences, and all workers have responsive preferences with quotas $(q_w)_{w \in W}$. For the symmetric case, the symmetric version of this proof applies. The following concepts will be useful. Given $\bar{I} \subseteq N$, $i \in \bar{I}$, and matching μ, the *set of partners available to i under* $(\bar{I}, \mu)$, denoted by $A_i(\mu)$,[12] is

$$A_i(\mu) = \mu(i) \cup \left\{ j \in P_i \cap \bar{I} \mid i \in \mathrm{Ch}_j(\mu(j) \cup i) \right\}.$$

In words, $A_i(\mu)$ is the set of agents that are either (i) matched to i under μ, or (ii) included in $\bar{I}$ and willing to block μ with i. Note that $\bar{I}$ is internally stable under a matching μ if and only if $\mu(i) = \mathrm{Ch}_i(A_i(\mu))$ for all $i \in \bar{I}$.

Now, given another matching μ', for each $f \in F$, we define $\tilde{A}_f(\mu, \mu')$ by

$$\tilde{A}_f(\mu, \mu') = \mu(f) \cup \left\{ w \in P_f \cap \bar{I} \mid f \in \mathrm{Ch}_w(\mu(w) \cup f) \cap \mathrm{Ch}_w(\mu'(w) \cup f) \right\}.$$

In words, $\tilde{A}_f(\mu, \mu')$ is the set of workers that are either (i) matched to f under μ, or (ii) included in $\bar{I}$ and willing to block μ and μ' with f.[13] For each $f \in F$, if there exists some $w \in \tilde{A}_f(\mu, \mu')$ such that (f, w) blocks μ, we say that the pair (f, w) is *firm-pointed for μ with respect to μ'*.

Now we prove Lemma 2. Suppose that I is internally stable under $\bar{\mu}$. Let $\bar{\imath} \notin I$ and $\bar{I} = I \cup \bar{\imath}$.

If there exists no $j \in P_{\bar{\imath}} \cap I$ such that $(\bar{\imath}, j)$ blocks $\bar{\mu}$, then it is clear that $\bar{I}$ is internally stable under $\mu_S \equiv \bar{\mu}$. Thus assume, for the rest of the proof, that there exists $j \in P_{\bar{\imath}} \cap I$ such that $(\bar{\imath}, j)$ blocks $\bar{\mu}$. □

Step 1 We match $\bar{\imath}$ with each agent in $\mathrm{Ch}_{\bar{\imath}}(A_{\bar{\imath}}(\bar{\mu})) \setminus \bar{\mu}(\bar{\imath})$ sequentially until $\bar{\imath}$ is matched to $\mathrm{Ch}_{\bar{\imath}}(A_{\bar{\imath}}(\bar{\mu}))$.[14] Let the final matching be μ. If $\bar{I}$ is internally stable under μ, then we set $\mu_S \equiv \mu$, completing the proof of Lemma 2. If $\bar{I}$ is not internally stable under μ, but $\mathrm{Ch}_f(\tilde{A}_f(\mu, \bar{\mu})) = \mu(f)$ for all $f \in F \cap \bar{I}$, then we skip Step 2, and proceed to Step 3. Otherwise, there exists $f \in F \cap \bar{I}$ such that $\mathrm{Ch}_f(\tilde{A}_f(\mu, \bar{\mu})) \neq \mu(f)$ and we proceed to Step 2.

Step 2 We match f with each worker in $\mathrm{Ch}_f(\tilde{A}_f(\mu, \bar{\mu})) \setminus \mu(f)$ sequentially until f is matched to $\mathrm{Ch}_f(\tilde{A}_f(\mu, \bar{\mu}))$. That is, we satisfy *firm-pointed* blocking pairs involving f with respect to $\bar{\mu}$ in this case. We iterate this process for all firms in $F \cap \bar{I}$ as long as there is a firm $f' \in F \cap \bar{I}$ with $\mathrm{Ch}_{f'}(\tilde{A}_{f'}(\mu', \bar{\mu})) \neq \mu'(f')$,

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[11] Note that there always exists an internally stable set of agents for any matching. An empty set is an example.

[12] We suppress $\bar{I}$.

[13] In the rest of the proof, we will take $\mu' \equiv \bar{\mu}$, the initial matching.

[14] Since agents have substitutable preferences, each of these re-matchings can be executed as a pairwise blocking. Similar remarks apply for Steps 2 and 3.

where μ' is the current matching. We refer to each such iteration as round $r = 1, 2, \ldots$, in the order of execution. Let μ^r denote the matching at the beginning of round r of Step 2 (for example, μ^1 is the matching obtained at the end of Step 1 and the beginning of round 1 of Step 2). We claim that in each round of this execution, no firm dumps any worker, implying that each worker is made weakly better off. More specifically,

Claim 1 *For all rounds r of Step 2, and all $f \in F \cap \bar{I}$, $\mu^r(f) \subseteq \mathrm{Ch}_f(\tilde{A}_f(\mu^r, \bar{\mu}))$, implying that no firm dumps any worker in any round of Step 2.*

Proof We prove the claim by induction. □

- Let $r = 1$. If $\bar{\imath} \in F \cap \bar{I}$, then by construction, blocking pairs that include $\bar{\imath}$ are satisfied in $\bar{I}$ in Step 1, implying $\mu^1(\bar{\imath}) = \mathrm{Ch}_{\bar{\imath}}(\tilde{A}_{\bar{\imath}}(\mu^1, \bar{\mu}))$. Let $f \in \left(F \cap \bar{I}\right) \backslash \bar{\imath}$. Note that in Step 1, workers in $\bar{\mu}(f)$ may dump f, but no worker is newly matched with f, implying $\mu^1(f) \subseteq \bar{\mu}(f)$. Therefore,

$$\begin{aligned}\tilde{A}_f(\mu^1, \bar{\mu}) &= \mu^1(f) \cup \left\{w \in W \cap \bar{I} \mid f \in \mathrm{Ch}_w(\mu^1(w) \cup f) \cap \mathrm{Ch}_w(\bar{\mu}(w) \cup f)\right\} \\ &\subseteq \mu^1(f) \cup \left\{w \in W \cap \bar{I} \mid f \in \mathrm{Ch}_w(\bar{\mu}(w) \cup f)\right\} \\ &\subseteq \bar{\mu}(f) \cup \left\{w \in W \cap \bar{I} \mid f \in \mathrm{Ch}_w(\bar{\mu}(w) \cup f)\right\} \\ &= A_f(\bar{\mu}).\end{aligned}$$

  Note that if a worker $w \in W \cap \bar{I}$ dumps f in Step 1, then $f \notin Ch_w\left(\mu^1(w) \cup f\right)$, which in turn implies that $w \notin \tilde{A}_f(\mu^1, \bar{\mu})$. This, together with $\mu^1(f) \subseteq \bar{\mu}(f)$, imply that

$$\mu^1(f) = \bar{\mu}(f) \cap \tilde{A}_f(\mu^1, \bar{\mu}).$$

  Moreover, since I is internally stable under $\bar{\mu}$, we have $\bar{\mu}(f) = \mathrm{Ch}_f(A_f(\bar{\mu}))$. Therefore,

$$\begin{aligned}\mu^1(f) &= \mathrm{Ch}_f(A_f(\bar{\mu})) \cap \tilde{A}_f(\mu^1, \bar{\mu}) \\ &\subseteq \mathrm{Ch}_f(\tilde{A}_f(\mu^1, \bar{\mu})), \qquad (1)\end{aligned}$$

  where the last set inclusion follows from $\tilde{A}_f(\mu^1, \bar{\mu}) \subseteq A_f(\bar{\mu})$ and substitutability of preferences of f. Equation 1 implies that no worker is dumped in round 1.
- Let $r \geq 1$. In the inductive step, assume that for all $f \in F \cap \bar{I}$, $\mu^r(f) \subseteq \mathrm{Ch}_f(\tilde{A}_f(\mu^r, \bar{\mu}))$, implying that no worker is dumped in round r. Let $f' \in F \cap \bar{I}$ be the firm which is satisfied in round r. Since round r dictates f' to be matched with $\mathrm{Ch}_{f'}(\tilde{A}_{f'}(\mu^r, \bar{\mu})) = \mathrm{Ch}_{f'}(\tilde{A}_{f'}(\mu^{r+1}, \bar{\mu}))$, we have $\mu^{r+1}(f') = \mathrm{Ch}_{f'}(\tilde{A}_{f'}(\mu^{r+1}, \bar{\mu}))$. Let $f \in \left(F \cap \bar{I}\right) \backslash f'$. In round r, workers in $\mu^r(f)$ may dump f, but no worker is newly matched with f, implying that $\mu^{r+1}(f) \subseteq \mu^r(f)$. Moreover, by the inductive assumption, no worker is dumped in

round r. Suppose $f \in \mathrm{Ch}_w\left(\mu^{r+1}(w) \cup f\right)$ for some $w \in W \cap \bar{I}$. Worker w has responsive preferences, implying that her preferences are also substitutable. Therefore, by Lemma 1, $f \in \mathrm{Ch}_w(\mu^r(w) \cup f)$. This, together with $\mu^{r+1}(f) \subseteq \mu^r(f)$, imply that

$$\begin{aligned}\tilde{A}_f(\mu^{r+1}, \bar{\mu}) &= \mu^{r+1}(f) \cup \left\{w \in W \cap \bar{I} \mid f \in \mathrm{Ch}_w(\mu^{r+1}(w) \cup f) \cap \mathrm{Ch}_w(\bar{\mu}(w) \cup f)\right\} \\ &\subseteq \mu^{r+1}(f) \cup \left\{w \in W \cap \bar{I} \mid f \in \mathrm{Ch}_w(\mu^{r}(w) \cup f) \cap \mathrm{Ch}_w(\bar{\mu}(w) \cup f)\right\} \\ &\subseteq \mu^{r}(f) \cup \left\{w \in W \cap \bar{I} \mid f \in \mathrm{Ch}_w(\mu^{r}(w) \cup f) \cap \mathrm{Ch}_w(\bar{\mu}(w) \cup f)\right\} \\ &= \tilde{A}_f(\mu^r, \bar{\mu}).\end{aligned}$$

Note that if a worker $w' \in W \cap \bar{I}$ dumps f in round $r+1$, then $f \notin \mathrm{Ch}_{w'}\left(\mu^{r+1}\left(w'\right) \cup f\right)$, which in turn implies $w' \notin \tilde{A}_f(\mu^{r+1}, \bar{\mu})$. This, together with $\mu^{r+1}(f) \subseteq \mu^r(f)$, imply that

$$\mu^{r+1}(f) = \mu^r(f) \cap \tilde{A}_f(\mu^{r+1}, \bar{\mu}).$$

Moreover, by the inductive assumption, $\mu^r(f) \subseteq \mathrm{Ch}_f(\tilde{A}_f(\mu^r, \bar{\mu}))$. Therefore,

$$\begin{aligned}\mu^{r+1}(f) &\subseteq \mathrm{Ch}_f(\tilde{A}_f(\mu^r, \bar{\mu})) \cap \tilde{A}_f(\mu^{r+1}, \bar{\mu}) \\ &\subseteq \mathrm{Ch}_f(\tilde{A}_f(\mu^{r+1}, \bar{\mu})),\end{aligned} \tag{2}$$

where the last set inclusion follows from $\tilde{A}_f(\mu^{r+1}, \bar{\mu}) \subseteq \tilde{A}_f(\mu^r, \bar{\mu})$ and substitutability of preferences of f. Equation 2 also shows that worker is dumped in round $r+1$ of Step 2, completing the induction. □

Since at least one worker in $W \cap \bar{I}$ is made strictly better off and no worker is made worse off in each round, the iterative process in Step 2 eventually stops. Let the final matching be μ. If $\mathrm{Ch}_w(A_w(\mu)) = \mu(w)$ for all $w \in W \cap \bar{I}$, then we skip Step 3. In this case, $\bar{I}$ is internally stable under $\mu_S \equiv \mu$, and this completes the proof of Lemma 2. Otherwise, there exists $w \in W \cap \bar{I}$ such that $\mathrm{Ch}_w(A_w(\mu)) \neq \mu(w)$. We proceed to Step 3.

Step 3 We match w to her most preferred firm in $\mathrm{Ch}_w(A_w(\mu)) \setminus \mu(w)$. We iterate this process for all workers in $W \cap \bar{I}$ as long as there is a worker $w' \in W' \cap \bar{I}$ with $\mathrm{Ch}_{w'}(A_{w'}(\mu')) \neq \mu'(w')$, where μ' is the current matching. We refer to each such iteration as round $r = 1, 2, \ldots$, in the order of execution. Let $X^r \subseteq W \cap \bar{I}$ be the set of workers who have been dumped at least once, by the beginning of round r in Steps 1 and 3 (for example, since no worker is dumped in Step 2, X^1 is the set of workers in $\bar{I}$ dumped in Step 1). Let (f^r, w^r) be the blocking pair satisfied in round r. Let μ^r denote the matching at the beginning of round r of Step 3 (for example, μ^1 is the matching obtained at the end of Step 2 and at the beginning of round 1 of Step 3). We claim that in each round of this execution, no worker dumps any firm, implying that each firm is made weakly better off.

Claim 2 *No worker dumps any firm in any round of Step 3.*

Proof We prove the claim by induction. □

- Let $r = 1$. First we show that $w^1 \in X^1$, by contradiction. Suppose $w^1 \in \left(W \cap \bar{I}\right) \backslash X^1$. This, together with Lemma 1 and $f^1 \in \mathrm{Ch}_{w^1}\left(\mu^1\left(w^1\right) \cup f^1\right)$, imply that $f^1 \in \mathrm{Ch}_{w^1}\left(\bar{\mu}\left(w^1\right) \cup f^1\right)$. Therefore, $w^1 \in \tilde{A}_{f^1}(\mu^1, \bar{\mu})$, implying that $\left(f^1, w^1\right)$ is a firm-pointed blocking pair in $\bar{I}$ for μ^1. This contradicts the fact that Step 2 has already stopped. Thus, $w^1 \in X^1$.
  Second we show that $\left|\mu^1\left(w^1\right)\right| < q_{w^1}$, by contradiction. Suppose that $\left|\mu^1\left(w^1\right)\right| = q_{w^1}$. Hence, by Observation 1, when $\left(f^1, w^1\right)$ is satisfied, one firm will be dumped. Let $\hat{f}^1$ be this firm, that is, $\hat{f}^1 = \mu^1\left(w^1\right) \backslash \mathrm{Ch}_{w^1}\left(\mu^1\left(w^1\right) \cup f^1\right)$. Since $w^1 \in X^1$, worker w^1 is dumped in Step 1, and since $\left|\mu^1\left(w^1\right)\right| = q_{w^1}$, she fills her quota back in Step 2. Let $\hat{F}^1$ be the set of firms such that for all $f \in \hat{F}^1$, the pair $\left(f, w^1\right)$ is satisfied in Step 2. Since all pairs satisfied in Step 2 are firm-pointed pairs, then $f \in \mathrm{Ch}_{w^1}\left(\bar{\mu}\left(w^1\right) \cup f\right)$ for all $f \in \hat{F}^1$. Since $\left(f^1, w^1\right)$ is not firm-pointed, then $f^1 \notin \mathrm{Ch}_{w^1}\left(\bar{\mu}\left(w^1\right) \cup f^1\right)$. Hence, by responsiveness of preferences of w^1, for all $f \in \hat{F}^1$, we have $f \succ_{w^1} f^1$. Since $\left(f^1, w^1\right)$ blocks μ^1, we have $f^1 \succ_{w^1} \hat{f}^1$. The previous two statements imply that $\hat{f}^1 \notin \hat{F}^1$. Since $\hat{f}^1$ is not satisfied in Step 2, we also have that $\hat{f}^1 \in \bar{\mu}\left(w^1\right)$ or $\hat{f}^1$ is matched to w^1 in Step 1. In either case, by responsiveness of preferences of w^1, we have $f^1 \in \mathrm{Ch}_{w^1}\left(\bar{\mu}\left(w^1\right) \cup f^1\right)$, implying that $\left(f^1, w^1\right)$ is a firm-pointed blocking pair in $\bar{I}$ for μ^1, a contradiction. We showed that $\left|\mu^1\left(w^1\right)\right| < q_{w^1}$, implying, together with Observation 1, that no firm is dumped, and each firm is made weakly better off in round 1.
- Let $r \geq 1$. In the inductive step, assume that for all rounds s such that $1 \leq s < r$, no firm is dumped in round s, and each firm is made weakly better off in round s.
  First we show that $w^r \in X^r$, by contradiction. Suppose $w^r \in \left(W \cap \bar{I}\right) \backslash X^r$. Therefore, worker w^r is made weakly better off throughout Step 1, Step 2, and until round r of Step 3. By the inductive assumption, each firm is made weakly better off in Step 3 until round r. These two statements, together with Lemma 1 and the fact that (f^r, w^r) blocks μ^r, imply that (i) $f^r \in \mathrm{Ch}_{w^r}\left(\bar{\mu}\left(w^r\right) \cup f^r\right)$, $f^r \in \mathrm{Ch}_{w^r}\left(\mu^1\left(w^r\right) \cup f^r\right)$, and (ii) $w^r \in \mathrm{Ch}_{f^r}\left(\mu^1\left(f^r\right) \cup w^r\right)$. Therefore, $w^r \in \tilde{A}_{f^r}(\mu^1, \bar{\mu})$ and (f^r, w^r) blocks μ^1, meaning that (f^r, w^r) is a firm-pointed blocking pair in $\bar{I}$ for μ^1. This contradicts the fact that Step 2 has already stopped. Thus, $w^r \in X^r$.
  Second we show that $\left|\mu^r\left(w^r\right)\right| < q_{w^r}$, by contradiction. Suppose that $\left|\mu^r\left(w^r\right)\right| = q_{w^r}$. Hence, by Observation 1, when (f^r, w^r) is satisfied, one firm will be dumped. Let $\hat{f}^r$ be this firm, that is, $\hat{f}^r = \mu^r\left(w^r\right) \backslash \mathrm{Ch}_{w^r}\left(\mu^r\left(w^r\right) \cup f^r\right)$. Since $w^r \in X^r$, worker w^r is dumped in Step 1 or earlier rounds of Step 3, and since $\left|\mu^r\left(w^r\right)\right| = q_{w^r}$, she fills her quota back in Step 2 or earlier rounds of Step 3. Let $\tilde{F}^r \subseteq \mu^r\left(w^r\right)$ be the set of firms such that for all $f \in \tilde{F}^r$, the pair (f, w^r) is satisfied at some earlier round $s < r$ of Step 3, and let

$\hat{F}^r \subseteq \mu^r(w^r) \setminus \tilde{F}^r$ be the set of firms such that for all $f \in \hat{F}^r$, the pair (f, w^r) is satisfied in Step 2. We will show that for all $f \in \tilde{F}^r \cup \hat{F}^r$, $f \succ_{w^r} f^r$, in two steps:

Claim 2A For all $f \in \tilde{F}^r$, $f \succ_{w^r} f^r$.

Proof Let $f \in \tilde{F}^r$. Suppose (f, w^r) is satisfied in round $s < r$ of Step 3. By the inductive assumption, f^r is made weakly better off in Step 3 until round r. This, together with Lemma 1 and the fact that $w^r \in \mathrm{Ch}_{f^r}(\mu^r(f^r) \cup w^r)$, imply that $w^r \in \mathrm{Ch}_{f^r}(\mu^s(f^r) \cup w^r)$. Hence, $f^r \in A_{w^r}(\mu^s)$. By construction of Step 3, worker w^r is matched with her most preferred firm among the ones with whom she is blocking in round s. Since f is chosen instead of f^r in round s, we have $f \succ_{w^r} f^r$. □

Claim 2B For all $f \in \hat{F}^r$, $f \succ_{w^r} f^r$.

Proof By the inductive assumption, each firm is made weakly better off in Step 3 until round r. This, together with Lemma 1 and the fact that $w^r \in \mathrm{Ch}_{f^r}(\mu^r(f^r) \cup w^r)$, imply that $w^r \in \mathrm{Ch}_{f^r}\left(\mu^1(f^r) \cup w^r\right)$. Since (f^r, w^r) is not a firm-pointed blocking pair for μ^1 (otherwise Step 2 would have not stopped), we have

$$f^r \notin \mathrm{Ch}_{w^r}\left(\mu^1\left(w^r\right) \cup f^r\right) \text{ or } f^r \notin \mathrm{Ch}_{w^r}\left(\bar{\mu}\left(w^r\right) \cup f^r\right).$$

- *Case A* $f^r \notin \mathrm{Ch}_{w^r}\left(\mu^1(w^r) \cup f^r\right)$: For all $f \in \hat{F}^r$, we have $f \in \mu^1(w^r)$. By responsiveness of preferences of w^r, for all $f \in \hat{F}^r$, we have $f \succ_{w^r} f^r$.
- *Case B* $f^r \notin \mathrm{Ch}_{w^r}(\bar{\mu}(w^r) \cup f^r)$: Since only firm-pointed blocking pairs are satisfied through Step 2, for all $f \in \hat{F}^r$, $f \in \mathrm{Ch}_{w^r}(\bar{\mu}(w^r) \cup f)$. By responsiveness of preferences of w^r, for all $f \in \hat{F}^r$, we have $f \succ_{w^r} f^r$. □

Since (f^r, w^r) blocks μ^r, we have $f^r \succ_{w^r} \hat{f}^r$. By Claims 2A and 2B, for all $f \in \tilde{F}^r \cup \hat{F}^r$, $f \succ_{w^r} f^r$. The last two statements imply that $\hat{f}^r \notin \tilde{F}^r \cup \hat{F}^r$. Since $(\hat{f}^r, w^r)$ is not satisfied in Steps 2 or 3, and $\hat{f}^r$ is kept as a partner by w^r until round r, we have that $\hat{f}^r \in \bar{\mu}(w^r)$ or $\hat{f}^r$ is matched at Step 1. In either case, by responsiveness of preferences of w^r, we have $f^r \in \mathrm{Ch}_{w^r}(\bar{\mu}(w^r) \cup f^r)$, and since $\hat{f}^r \in \mu^1(w^r)$, $f^r \in \mathrm{Ch}_{w^r}(\mu^1(w^r) \cup f^r)$. This, together with the fact that $w^r \in \mathrm{Ch}_{f^r}(\mu^1(f^r) \cup w^r)$, imply that (f^r, w^r) blocks μ^1, and $w^r \in \tilde{A}_f\left(\mu^1, \bar{\mu}\right)$. Therefore, (f^r, w^r) is a firm-pointed blocking pair in $\bar{I}$ for μ^1, contradicting the fact that Step 2 has already stopped. Thus, $|\mu^r(w^r)| < q_{w^r}$, implying, together with Observation 1, that no firm is dumped, and each firm is made weakly better off in round r, completing the induction. □

Since each firm in $F \cap \bar{I}$ is made weakly better off and one firm in $F \cap \bar{I}$ is made strictly better off in each round, the iterative process in Step 3 eventually stops at a matching μ_S.

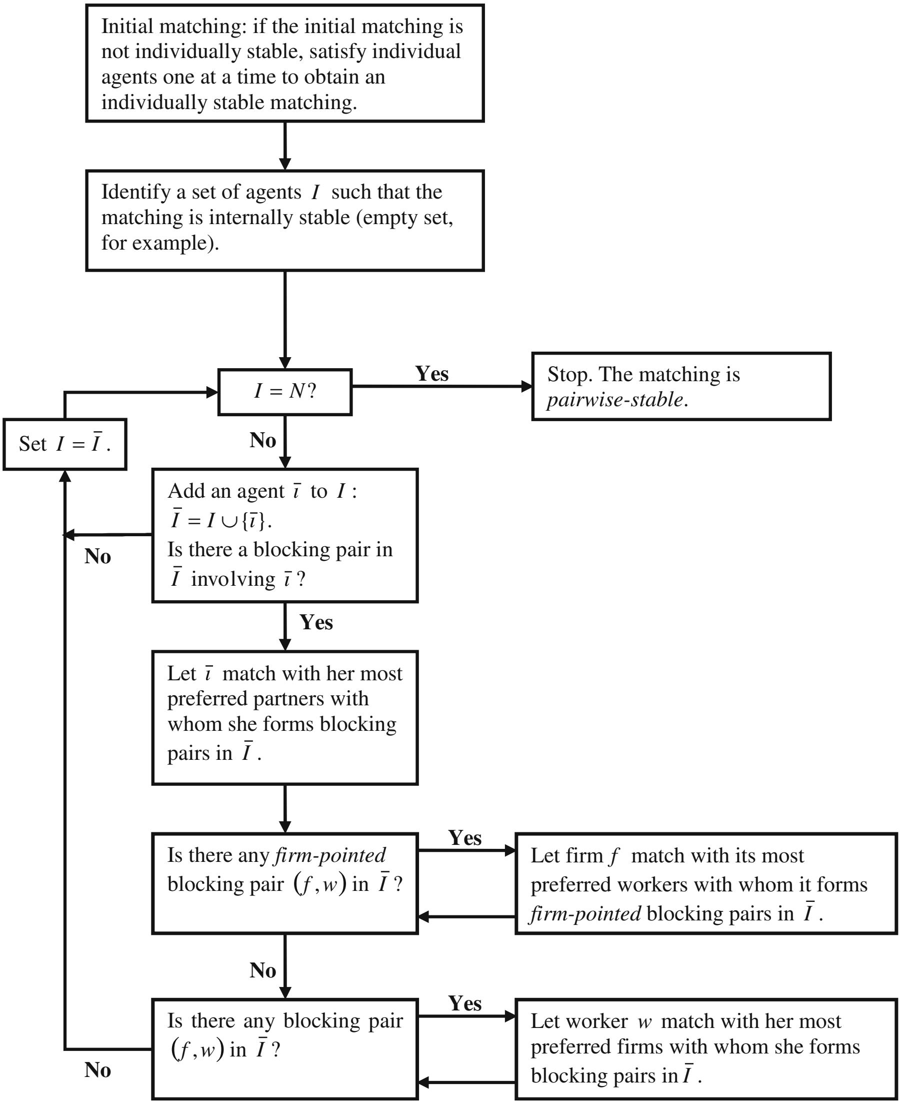

Fig. 1 Starting from a pairwise-unstable matching, construction of a convergent blocking path to a pairwise-stable matching

Since Step 3 stops, $\bar{I}$ is internally stable under μ_S, completing the proof of Lemma 2. □

We outline the algorithm that proves Theorem 1 in Fig 1.

Next we show how the algorithm in the proof of Lemma 2 works with a simple example:

Example Let $F = \{f_1, f_2, f_3, f_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Firms have substitutable preferences given by

$$
\begin{aligned}
&f_1 : w_1, w_3\\
&f_2 : w_2, w_4, w_3\\
&f_3 : w_3, w_4\\
&f_4 : \{w_1, w_2\}, w_1, w_2, \{w_3, w_4\}, w_3, w_4
\end{aligned}
$$

where the notational convention is that, each row represents the acceptable sets of partners in order of preference: For instance, f_1 prefers w_1 to w_3, and w_1 and w_3 are the only acceptable workers. Workers have responsive preferences. Let $q_{w_3} = 2, q_{w_1} = q_{w_2} = q_{w_4} = 1$, and their preferences over individual firms be given by

$$
\begin{aligned}
&w_1 : f_4, f_1\\
&w_2 : f_4, f_1\\
&w_3 : f_4, f_1, f_2, f_3\\
&w_4 : f_4, f_2, f_3
\end{aligned}
$$

The initial matching $\bar{\mu}$ is given by

$$
\begin{aligned}
&\bar{\mu}(f_1) = w_1\\
&\bar{\mu}(f_2) = w_2\\
&\bar{\mu}(f_3) = w_3\\
&\bar{\mu}(f_4) = \{w_3, w_4\}
\end{aligned}
$$

Note that $\bar{\mu}$ is individually stable, but it is not pairwise-stable as, for example, the pair (f_4, w_1) blocks $\bar{\mu}$. Now we follow our algorithm to enlarge the internally stable set of agents. Let $I = \{f_1, f_2, f_3, w_1, w_2, w_3, w_4\}$. It is easy to see that I is internally stable under $\bar{\mu}$.[15] Now let f_4 join the set, so $\bar{I} = I \cup f_4$, initiating the algorithm.

Step 1 Firm f_4 is matched to its most preferred available workers $\{w_1, w_2\}$ and dumps w_3 and w_4. As they are matched to f_4, w_1 and w_2 dump f_1 and f_2, respectively.

Step 2 The firm-pointed blocking pairs are satisfied. Consider f_1. Although f_1 prefers w_1 most, w_1 is matched to her most preferred firm f_4. The next preferred worker by f_1 is w_3. Worker w_3 is available, since she prefers f_1 to her current *unique* partner f_3, who was also matched to w_3 under $\bar{\mu}$, implying, together with responsiveness of w_3's preferences, that (f_1, w_3) is a firm-pointed blocking pair. Therefore, blocking pair (f_1, w_3) is satisfied. Since $q_{w_3} = 2$, worker w_3 does not dump any firm. Consider f_2, which is currently unmatched. Worker w_3 is an acceptable partner of f_2. Firm f_2 is worker w_3's favorite partner. Hence, by responsiveness of w_3's preferences, pair (f_2, w_3)

---

[15] Although the way we found I is somewhat arbitrary in this example, one can always find an internally stable set. For instance, the empty set is internally stable for any matching.

is a firm-pointed blocking pair. Pair (f_2, w_3) is satisfied, while f_3 is dumped by w_3. There are two remaining blocking pairs, (f_2, w_4) and (f_3, w_4). Since f_2 and f_3 are less preferred by w_4 to f_4, who used to be matched to w_4 under $\bar{\mu}$, and $q_{w_2} = 1$, neither of them is a firm-pointed blocking pair. Therefore, neither one is satisfied in Step 2. Thus, Step 2 stops. Note that no worker is dumped in Step 2, while firm f_3 is dumped.

Step 3 The remaining blocking pairs are satisfied. Since f_2 is preferred to f_3 by w_4, blocking pair (f_2, w_4) is satisfied, and w_3 is dumped by f_2. Finally, blocking pair (f_3, w_3) is satisfied. Note that no firm is dumped in Step 3, while worker w_3 is dumped.

The algorithm stops, as there is no blocking pair in $\bar{I}$.

The final matching μ is given by

$$\begin{aligned} \mu(f_1) &= w_3 \\ \mu(f_2) &= w_4 \\ \mu(f_3) &= w_3 \\ \mu(f_4) &= \{w_1, w_2\} \end{aligned}$$

Since $\bar{I} = F \cup W$, μ is pairwise-stable. □

We conclude our paper with the following three remarks:

Remark 1 Preferences of an agent are *categorywise-responsive*, if her potential partners are classified into disjoint categories such that her preferences are responsive with a quota in each category, and her preferences across categories are separable (Konishi and Ünver 2006). It is trivial to extend our theorem to the domain in which agents on one side have substitutable preferences and agents on the other have categorywise-responsive preferences. We can simply treat an agent with categorywise-responsive preferences as a combination of separate agents, one for each of her categories, and apply our construction.

Remark 2 For one-to-one matching, when $\mu_0 = \emptyset$ and W (or F) is the initial internally stable set, Roth and Vande Vate (1990) process coincides with the sequential version of the deferred acceptance algorithm (Gale and Shapley 1962) as introduced by McVitie and Wilson (1971). Similarly, in many-to-many matching, when $\mu_0 = \emptyset$ and we take W (F) as the initial internally stable set in the proof of Lemma 2, the process defined in the proof is equivalent to the sequential version of the firm-proposing (worker-proposing) deferred acceptance algorithm. In particular, Theorem 1 shows the existence of a pairwise-stable matching under the current assumptions.

Remark 3 Consider a stochastic process that, for each matching μ, assigns a positive probability to every blocking pair and individual of μ, chooses one randomly, and satisfies it to obtain a new matching. By Theorem 1, this process converges to a pairwise-stable matching in finite time with probability 1 for any initial matching.

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