ELSEVIER
Journal of Economic Behavior & Organization xxx (2007) xxx–xxx
JOURNAL OF
Economic Behavior
& Organization
www.elsevier.com/locate/econbase

p-Dominance and perfect foresight dynamics

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Received 12 March 2006; received in revised form 5 July 2007; accepted 7 July 2007

Abstract

We investigate stability of **p**-dominant equilibria under perfect foresight dynamics.We show that a strict **p**-dominant equilibrium with $\sum_i p_i < 1$ is globally accessible and absorbing in perfect foresight dynamics. We also investigate robustness and extensions of this result. We apply our proof method to games with u-dominant equilibria and unanimity games.

JEL classification: C72; C73

Keywords: Equilibrium selection; Perfect foresight dynamics; **p**-Dominance

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1. Introduction

Perfect foresight dynamics (Matsui and Matsuyama, 1995) is a model of rational and forward-looking individuals. Facing inertia of action revisions, agents in large populations take best responses to the discounted time-average of the action distributions from the present to the future.

Under this dynamics, even a strict Nash equilibrium can be upset by a consistent belief that the society will move away from the equilibrium. This feature makes it possible to use perfect foresight dynamics for equilibrium selection. Specifically, Oyama et al. (2006, OTH henceforth) show that the strict monotone-potential maximizer in a strict monotone-potential game is a *globally accessible* and *absorbing* state if the original game or the strict monotone-potential function is supermodular. That is, there exists a path converging to the state, no matter how far the initial state of the society is, and no path can escape from the state once the path is close to the state. For an n-player asymmetric game and $\mathbf{p} = (p_1, \ldots, p_n)$, an action profile a^* is a (strict) **p**-dominant equilibrium of the game if, for every player i, action a_i^* is a (unique) best response to any belief putting probability at least (or more than) p_i that other players take a^*_{-i}. Since (strict) monotone-potential maximization with supermodular monotone-potential functions generalizes (strict) **p**-dominance with $\sum_{i=1}^n p_i < 1$, OTH's result implies that strict **p**-dominant equilibria with $\sum_i p_i < 1$ are globally accessible and absorbing.

We present an alternative proof for global accessibility and absorption of strict **p**-dominant equilibria with $\sum_i p_i < 1$. For example, global accessibility with zero subjective discount rates is shown as follows. Let a^* be a **p**-dominant equilibrium with $\sum_i p_i < 1$ and x be the initial state. Our proof begins with showing the following: there exists $\mathbf{T} = (T_1, \ldots, T_n)$ such that each agent who receives revision opportunity after T_i puts probability at least p_i, under

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doi:10.1016/j.jebo.2007.07.002

Please cite this article in press as: Kojima, F., Takahashi, S., **p**-Dominance and perfect foresight dynamics, Journal of Economic Behavior and Organization (2007), doi:10.1016/j.jebo.2007.07.002

the effectively discounted belief, that the opponent in population j will have received revision opportunity after T_j for every $j \neq i$. Given such $\mathbf{T}$, we consider the restricted set $\Phi(x, \mathbf{T})$ of paths originating at x where agents in population i are required to take a_i^* if they receive revision opportunities after T_i. $\Phi(x, \mathbf{T})$ is nonempty, convex and compact. By the construction of $\mathbf{T}$ and the definition of $\mathbf{p}$-dominance, taking a_i^* after T_i is a best response for agents in population i if agents in population j take a_j^* after T_j for every $j \neq i$. Therefore, the best response correspondence from $\Phi(x, \mathbf{T})$ to itself is a nonempty-valued correspondence. By the fixed point theorem, there is a fixed point of the correspondence, which is a perfect foresight path from x to a^*.

Kajii and Morris (1997) consider an alternative method of equilibrium selection. They say that a Nash equilibrium of a complete information game is robust to incomplete information if every incomplete information game with payoffs almost always given by the complete information game has an equilibrium whose observed behavior is close enough to the Nash equilibrium. They show that any $\mathbf{p}$-dominant equilibrium with $\sum_i p_i < 1$ is robust to incomplete information. Their proof goes as follows. Let Ω be the countable set of states, P be a common prior on Ω, and $\mathcal{Q}_i$ be player i's information partition of Ω. For $\omega \in \Omega$, $Q_i(\omega)$ denotes the element of $\mathcal{Q}_i$ that contains ω. An event $E \subseteq \Omega$ is $\mathbf{p}$-believed at $\omega \in \Omega$ if each player i believes E at least with probability p_i conditional on $Q_i(\omega)$, that is, $P[E|Q_i(\omega)] \geq p_i$ for every i. Let $B_*^{\mathbf{p}}(E)$ be the set of states at which E is $\mathbf{p}$-believed. An event E is $\mathbf{p}$-evident if $E \subseteq B_*^{\mathbf{p}}(E)$, that is, if it is $\mathbf{p}$-believed whenever it is true. Kajii and Morris (1997) show the *critical path result*: if $\sum_i p_i < 1$, then for any event E with high ex ante probability, there exists a $\mathbf{p}$-evident subset F of E with high ex ante probability. Take E as the event that payoffs are given by those in the complete information game. Since the incomplete information game is close enough to the complete information game, $P[E]$ is close enough to one. By the critical path result, there exists a $\mathbf{p}$-evident subset F of E such that $P[F]$ is close enough to one. Then Kajii and Morris consider a restricted set of strategies in which each player i plays a_i^* at any state in F. By the construction of F and the definition of $\mathbf{p}$-dominance, taking a_i^* in F is a best response for player i if player j takes a_j^* in F for every $j \neq i$. Therefore the best response correspondence defined on this restricted strategy set is nonempty-valued. Thus there is a fixed point of this best response correspondence, which is an equilibrium in the incomplete information game. In this equilibrium, the ex ante probability that agents play a^* in the equilibrium is close enough to one.

The two proofs are similar to each other. The product of intervals, $\prod_{i=1}^{n}[T_i, \infty)$, in our proof has the property that after T_i, every agent in population i puts probability at least p_i that her opponents' most recent revisions occured within this set. This corresponds to the $\mathbf{p}$-evident set F in Kajii and Morris' proof. In both proofs, the best response correspondence defined on the restricted strategy set is nonempty-valued, and hence has a fixed point, which corresponds to an equilibrium of the incomplete information game and a perfect foresight path under the perfect foresight dynamics, respectively. Actually, the parallelism between the robustness approach and our result is more than a coincidence. As Takahashi (in press) points out, the perfect foresight dynamics can be regarded as a static incomplete information game where player i's type in the incomplete information game corresponds to time when an agent in population i receives a revision opportunity. A strategy profile is a Bayesian Nash equilibrium in the incomplete information game if and only if it induces a perfect foresight path in the perfect foresight dynamics.

Despite the aforementioned similarity, there is an important difference between our approach and Kajii and Morris (1997). In Kajii and Morris (1997), the critical path result implies that a $\mathbf{p}$-evident subset of a high ex ante probability event has a high ex ante probability, whereas our corresponding result shows the existence of finite $\mathbf{T}$ but is silent about the "ex ante probability" of the set $\prod_i[T_i, \infty)$. This is because, as Takahashi points out, in the perfect foresight dynamics interpreted as an incomplete information game, "types" are distributed according to an improper distribution. Therefore, it is impossible to obtain a result concerning the ex ante probability of $\prod_i[T_i, \infty)$, and we only have that this set is "large enough" in the sense that each T_i is finite. We introduce a technique of random ordering to cope with this difference. As explained in Section 3.1, we can associate random ordering with beliefs under the perfect foresight dynamics with zero subjective discount rates. This fact and the simple structure of random ordering enable us to find $\mathbf{T}$.

Our method of proof derives slightly stronger results than existing ones. We establish our results for not only positive but also zero and negative subjective discount rates. We also allow subjective discount rates and revision speeds to be heterogenous among populations.

We also investigate several issues concerning robustness and extensions of this result. First we consider rationalizable foresight dynamics defined by Matsui and Oyama (2006), which relaxes the perfect foresight assumption while retaining common knowledge of rationality. We show that a strict $\mathbf{p}$-dominant equilibrium is globally accessible and absorbing under rationalizable foresight dynamics. Then we consider the concept of (strict) $\mathbf{p}$-best response

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set, which is a set-valued extension of (strict) **p**-dominance proposed by Tercieux (2006). We see that a strict **p**-best response set with $\sum_i p_i < 1$ is globally accessible and absorbing as a set. Then we apply our proof method based on random ordering to other classes of games. Specifically, we rederive Kojima's (2006) result that any u-dominant equilibria is globally accessible and absorbing, and also investigate perfect foresight dynamics in unanimity games.

The parallelism between perfect foresight dynamics and the robustness approach to incomplete information has been present in the literature. The first result in the robustness approach is on **p**-dominance by Kajii and Morris. Next, Ui (2001) shows the robustness of potential maximizers in potential games. Then Morris and Ui (2005) extend Ui's proof to monotone-potentials and unify the previous results. By contrast, the literature on perfect foresight dynamics goes the other way around. First Hofbauer and Sorger (1999, 2002) show the stability results in potential games, and OTH extend these results to monotone-potential games. The **p**-dominance result has, however, not been proved without the potential method. This paper bridges the gap between the two strands of the literature by showing a counterpart of Kajii and Morris (1997) in perfect foresight dynamics.[1]

The rest of the paper proceeds as follows. Section 2 introduces perfect foresight dynamics and dynamic stability concepts. In Section 3 we prove our basic result. Section 4 establishes extensions of our basic result and also applies our method of order statistics to other classes of games. Section 5 concludes.

2. The model

For any nonempty finite set X, $\Delta(X) = \{f \in [0, 1]^X : \sum_{x \in X} f(x) = 1\}$ denotes the set of all probabilities on X. For two nonempty finite sets X and Y, $\Delta(X) \times \Delta(Y)$ is regarded as the set of independent probabilities on $X \times Y$, which is a subset of $\Delta(X \times Y)$. For any $f \in \Delta(X)$, $\mathrm{supp}(f) = \{x \in X : f(x) > 0\}$ is the support of f. For $x \in X$, $[x] \in \Delta(X)$ denotes the probability putting all the weight on x.

Consider a finite game $G = (N, A, u)$, where $N = \{1, \ldots, n\}$ is the set of players, A_i is the nonempty finite set of player i's actions, $A = \prod_{i=1}^n A_i$, $u_i : A \to \mathbb{R}$ is player i's payoff function, and $u = (u_1, \ldots, u_n)$. The domain of u_i is extended to $A_i \times \Delta(A_{-i})$ by the expected utility hypothesis, where $A_{-i} = \prod_{j \neq i} A_j$. We follow the convention that the subscript of $-i$ is used for profiles of player i's opponents. For any $x_{-i} \in \Delta(A_{-i})$, $\mathrm{br}_i(x_{-i})$ is the set of player i's pure action best responses to x_{-i}.

We denote by $\mathbb{R}_+$ the set of all nonnegative numbers. Let $\lambda_i > 0$ be the rate at which each agent in population i receives revision opportunity in a unit of time. Let $\phi_i(t)(a_i) \in [0, 1]$ be the fraction of agents in population i taking a_i at time t, $\phi_i(t) \in \Delta(A_i)$ be the action distribution in population i at time t, and $\phi_i : \mathbb{R}_+ \to \Delta(A_i)$ be the path of action distribution in population i. Suppose that agents in population i switch actions from $\phi_i(t)$ to $\alpha_i(t)$ when they can revise their actions at time t. Then ϕ_i satisfies the ordinal differential equation

$$\dot{\phi}_i(t) = \lambda_i(\alpha_i(t) - \phi_i(t)),$$

or

$$\phi_i(t) = \mathrm{e}^{-\lambda_i t}\phi_i(0) + \lambda_i \int_0^t \mathrm{e}^{\lambda_i(s-t)}\alpha_i(s)\,\mathrm{d}s$$

in the integral form. We say that a path ϕ_i is *induced* by α_i with revision speed λ_i if this relation holds for any $t \geq 0$, and $\phi = (\phi_1, \ldots, \phi_n)$ is i nduced by $\alpha = (\alpha_1, \ldots, \alpha_n)$ with revision speeds $\lambda = (\lambda_1, \ldots, \lambda_n)$ if ϕ_i is induced by α_i with revision speed λ_i for each $i \in N$. We say that ϕ is a *feasible path* if some measurable function α induces ϕ with revision speeds λ. Let $\Phi(x)$ be the set of all feasible paths from $\phi(0) = x$. Due to the Ascoli-Arzelà theorem, $\Phi(x)$ is compact with respect to the topology of uniform convergence on compact intervals. Let $\delta_i > -\lambda_i$ be the subjective discount rate of agents in population i. For a given feasible path ϕ_{-i}, revision speed λ_i and subjective discount rate δ_i,

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[1] We should mention Oyama's (2002) result. Working independently of Hofbauer and Sorger (2002), he shows that a strict (p, p)-dominant equilibrium with $p < 1/2$ is globally accessible and absorbing in a symmetric two-player game. In our context, however, we do not regard his result as a counterpart of Kajii and Morris, for it is not easy to generalize his proof to asymmetric n-player games.

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let

$$\pi_{-i}(t,\phi_{-i})(a_{-i}) := (\lambda_i+\delta_i)\int_t^{\infty} \mathrm{e}^{(\lambda_i+\delta_i)(t-s)}\left(\prod_{j\neq i}\phi_j(s)(a_j)\right)\,\mathrm{d}s$$

be the (normalized) effectively discounted probability that an agent in population i at time t expects her opponents to take a_{-i} until she gets the next revision opportunity. Note that $\pi_{-i}(t,\phi_{-i})\in\Delta(A_{-i})$, but $\pi_{-i}(t,\phi_{-i})\in\prod_{j\neq i}\Delta(A_j)$ may not hold in general. In other words, the time-averaged future behavior of player i's opponents may be correlated from player i's viewpoint at time t.

Definition 1. Fix revision speeds $\lambda=(\lambda_1,\ldots,\lambda_n)$ and subjective discount rates $\delta=(\delta_1,\ldots,\delta_n)$. A feasible path ϕ induced by α is a *perfect foresight path* on G if $\mathrm{supp}(\alpha_i(t))\subseteq \mathrm{br}_i(\pi_{-i}(t,\phi_{-i}))$ for any $i\in N$ and $t\geq 0$.

A path ϕ is a perfect foresight path from state x if and only if ϕ is a fixed point of the correspondence $\beta(x,\cdot)$ from $\Phi(x)$ to itself, where

$$\beta(x,\phi) := \{\psi\in\Phi(x) : \psi \text{ is induced by } \alpha, \mathrm{supp}(\alpha_i(t))\subseteq \mathrm{br}_i(\pi_{-i}(t,\phi_{-i})) \text{ for any } i\in N \text{ and } t\geq 0\}.$$

$\beta(x,\phi)$ is a nonempty and convex set, depending upper hemicontinuously on ϕ. Since $\Phi(x)$ is nonempty, compact, and convex, it follows from the Kakutani-Fan-Glicksberg fixed point theorem that there exists a perfect foresight path from x. See Oyama for details.

A microfoundation of the dynamics is as follows. There are n groups of infinitesimal and anonymous people, populations $1, 2, \ldots, n$, each of whose size is normalized to one. At every point in continuous time, one agent from each population is randomly chosen, and these n agents play the base game G. People cannot change their actions at every moment. Instead, each agent is committed to the same action for a while. Chances to change actions are given to individuals in population i by independent Poisson processes with arrival rate λ_i for each $i\in N$. Agents in population i switch actions from $\phi_i(t)$ to $\alpha_i(t)$ with speed λ_i. Thus the path ϕ_i of action distribution in population i is induced by α_i.

We assume that an agent, when given a revision opportunity, changes her action to maximize the expected value of her discounted payoff until she gets the next revision opportunity, given her expectation on the future path of behavior. Then the objective function of each agent at time t is given by $u_i(\cdot,\pi_{-i}(t,\phi_{-i}))$, given her belief ϕ_{-i} of the future path of behavior. Along a perfect foresight path, agents take best responses $\alpha(\cdot)$ to their expectations, and their expectations are the actual path itself. This is the reason the perfect foresight path is defined as a fixed point of the correspondence $\beta(x,\cdot)$.

We introduce the following stability concepts.

Definition 2. Let $x^*\in\prod_{i=1}^n\Delta(A_i)$.

(a) x^* is *globally accessible* if there exists a perfect foresight path from any state that converges to x^*.
(b) x^* is *absorbing* if there exists an open neighborhood $U\subseteq\prod_{i=1}^n\Delta(A_i)$ of x^* such that any perfect foresight path from any point of U converges to x^*.
(c) $x^*=[a^*]$ for some $a^*\in A$ is *linearly absorbing* if there exists an open neighborhood $U\subseteq\prod_{i=1}^n\Delta(A_i)$ of $[a^*]$ such that, for any $x\in U$, the path induced by $\alpha(t)=[a^*]$ is a unique perfect foresight path.

Global accessibility and absorption are defined by Matsui and Matsuyama. If x^* is globally accessible, then x^* can be reached under the dynamics whatever the initial state is. If x^* is absorbing, then no path can escape from it once it is reached. Linear absorption is a strengthening of absorption, requiring that a unique perfect foresight path approaches a^* as fast as possible. Absorption, on the other hand, allows multiplicity of perfect foresight paths and/or temporary deviation from x^*.

Although any globally accessible or absorbing state is a Nash equilibrium of G, the converse is not necessarily true. For example, Matsui and Matsuyama show that the risk-dominated equilibrium in a 2 × 2 coordination game is not absorbing because the equilibrium is upset by a belief that the society moves to the other risk-dominant equilibrium, and the belief is consistent with people's incentives. Therefore, the stability concepts might work as equilibrium selection.

Oyama gives a symmetric 3 × 3 game in which no state is globally accessible or absorbing. OTH give a 2 × 2 × 2 game that has two globally accessible states (and hence no absorbing state). Takahashi gives a symmetric 3 × 3 × 3

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game that has two absorbing states (and hence no globally accessible state). Therefore existence and uniqueness of a globally accessible or absorbing state does not hold in general, in which case equilibrium selection cannot be carried out based on perfect foresight dynamics. In the following sections, we give sufficient conditions for a state to be globally accessible or absorbing.

3. p-dominance

This section investigates the role of **p**-dominance in perfect foresight dynamics. First, we follow Morris et al. (1995) and Kajii and Morris (1997) to define (strict) **p**-dominance.

Definition 3. Let $a^* = (a_1^*, \ldots, a_n^*) \in A$ and $\mathbf{p} = (p_1, \ldots, p_n) \in [0, 1]^n$.

(a) a^* is a **p**-*dominant equilibrium* of G if, for any $i \in N$, $a_i^* \in \mathrm{br}_i(x_{-i})$ for any $x_{-i} \in \Delta(A_{-i})$ such that $x_{-i}(a_{-i}^*) \geq p_i$.
(b) a^* is a *strict* **p**-*dominant equilibrium* of G if, for any $i \in N$, $\{a_i^*\} = \mathrm{br}_i(x_{-i})$ for any $x_{-i} \in \Delta(A_{-i})$ such that $x_{-i}(a_{-i}^*) > p_i$.

A strict **p**-dominant equilibrium is always **p**-dominant if $p_i \neq 1$ for each i, and the converse is generically true. A (strict) **p**-dominant equilibrium is also (strict) **q**-dominant if $\mathbf{q} \geq \mathbf{p}$.[2]

The following is what we show in this section.

Theorem 1. *Let* $a^* \in A$.

(a) *If* a^* *is a* **p**-*dominant equilibrium of* G *with* $\sum_i p_i < 1$, *then, for any revision speeds* λ, *there exists* $\delta^* > 0$ *s* uch that $[a^*]$ *is globally accessible under subjective discount rates* δ *with* $-\lambda_i < \delta_i < \delta^*$ *for every* $i \in N$.
(b) *If* a^* *is a strict* **p**-*dominant equilibrium of* G *with* $\sum_i p_i < 1$, *then, for any revision speeds* λ, *there exists* $\delta_* < 0$ *s* uch that $[a^*]$ *is linearly absorbing under subjective discount rates* δ *with* $\delta_i > \delta_*$ *for every* $i \in N$.

Several remarks are in order. First, Theorem 1 extends results by Matsui and Matsuyama (1995) and Oyama (2002) to general asymmetric games. Matsui and Matsuyama discuss symmetric and asymmetric 2 × 2 games separately and show that the risk-dominant equilibrium is globally accessible and absorbing in each case. Oyama discusses two-player symmetric games and shows that a strict (p, p)-dominant equilibrium with $p < 1/2$ is, if exists, globally accessible and absorbing.

Second, a substantial part of Theorem 1 can be derived as a corollary of OTH's result. OTH show that the strict monotone-potential maximizer in a strict monotone-potential game is globally accessible and absorbing under sufficiently small discount rates if the original game or the strict monotone-potential function is supermodular. If the base game has a (strict) **p**-dominant equilibrium a^* with $\sum_i p_i < 1$, then there exists a supermodular (strict) monotone-potential function for G that attains its maximum at a^*. Thus OTH's result implies global accessibility and absorption of strict **p**-dominant equilibria with $\sum_i p_i < 1$.

Third, however, there are differences between this paper and OTH. One difference is in the method of proof. Second, the current proof method enables us to show slightly stronger results. Theorem 1 establishes global accessibility and linear absorption not only for positive discount rates but also for zero and negative discount rates that may be different among populations, whereas OTH assumes discount rates to be positive and homogeneous among populations.

Lastly, our theorem and its proof can be extended easily to the single-population model where n agents are drawn from one population and play a symmetric n-player game. The counterpart of Theorem 1 is the following, (strict) **p**-dominant equilibria with $p_i < 1/n$ for every $i \in N$ is globally accessible and absorbing if the discount rate is sufficiently small.

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[2] For $\mathbf{y}, \mathbf{z} \in \mathbb{R}^m$, we write $\mathbf{y} \geq \mathbf{z}$ if $y_i \geq z_i$ for every i.

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3.1. Random ordering

Let $(\zeta_i)_{i \in N}$ be independent random variables such that ζ_i is drawn from the exponential distribution with mean $1/\lambda_i$. For any nonempty subset S of N, let $\mathbf{T}_S = (T_i)_{i \in S} \in \mathbb{R}_+^S$. For $i \in S$ and k with $1 \le k \le \#S$, define $\rho_i^k(\mathbf{T}_S)$ as

$$\rho_i^k(\mathbf{T}_S) = \mathrm{Prob}(\#\{j \in S : \zeta_j + T_j \le \zeta_i + T_i\} = k),$$

and $\rho^k(\mathbf{T}_S) = (\rho_i^k(\mathbf{T}_S))_{i \in S}$.[3] $\rho_i^k(\mathbf{T}_S)$ is the probability that $\zeta_i + T_i$ is the k th smallest value among $(\zeta_j + T_j)_{j \in S}$. Note that ζ_i is drawn from a continuous distribution, and hence no tie occurs with positive probability.

A relevance of $\rho_i^k(\mathbf{T}_S)$ to our dynamics is given as follows. Consider feasible path ϕ with initial state $[a]$ and induced by α, which is defined by

$$\alpha_i(t) = \begin{cases} [a_i] & (t < T_i), \\ [b_i] & (t \ge T_i), \end{cases}$$

where $a_i \ne b_i$ for any $i \in S$ and $a_i = b_i$ for any $i \in N \setminus S$. In other words, ϕ is a feasible path such that agents in population i take a_i if they get revision opportunities before T_i and then take b_i after T_i. Then the fraction of agents from population $j \ne i$ taking action b_j at time t is $\max(0, 1 - \mathrm{e}^{\lambda_j(-t+T_j)})$. Thus, for any $i \in S$ and any $S' \subseteq S \setminus \{i\}$, the probability that an agent from population i matches with an $n-1$-tuple of agents who play b_j if and only if $j \in S'$ is

$$\prod_{j \in S'} \max(0, 1 - \mathrm{e}^{\lambda_j(-t+T_j)}) \prod_{j \in S \setminus \{i\} \setminus S'} \min(1, \mathrm{e}^{\lambda_j(-t+T_j)}).$$

Thus, the "discounted" probability at time T_i that an agent from population i plays against such opponents in the future along ϕ, $\pi_{-i}(T_i, \phi_{-i})(a_{N \setminus \{i\} \setminus S'}, b_{S'})$, is

$$\lambda_i \int_{T_i}^{\infty} \mathrm{e}^{\lambda_i(T_i - t)} \left(\prod_{j \in S'} \max(0, 1 - \mathrm{e}^{\lambda_j(-t+T_j)}) \prod_{j \in S \setminus \{i\} \setminus S'} \min(1, \mathrm{e}^{\lambda_j(-t+T_j)}) \right) \mathrm{d}t$$

under zero discount rates $\delta = (0, \ldots, 0)$. This is equal to the probability that $\zeta_j + T_j \le \zeta_i + T_i$ for $j \in S'$ and $\zeta_j + T_j > \zeta_i + T_i$ for $j \in S \setminus \{i\} \setminus S'$.

Summing up the above probability for all S' with $\#S' = k - 1$, we have the discounted probability at time T_i that an agent from population i plays against $k - 1$ opponents playing according to b in the future along ϕ, which is equal to $\rho_i^k(\mathbf{T}_S)$ by definition.

Lemma 1. *Let $\emptyset \ne S \subseteq N$ and $\mathbf{T}_S \in \mathbb{R}_+^S$.*

(a) $\sum_{i \in S} \rho_i^k(\mathbf{T}_S) = 1$ *for any k with $1 \le k \le \#S$.*
(b) $\rho_i^1(\mathbf{T}_S)$ *is decreasing in T_i and increasing in T_j for $j \in S \setminus \{i\}$; $\rho_i^{\#S}(\mathbf{T}_S)$ is increasing in T_i and decreasing in T_j for $j \in S \setminus \{i\}$.*
(c) *If $\lambda_i = \lambda_j$ for every $i, j \in S$, then $\sum_{i \in S} \rho_i^1(\mathbf{T}_S)/\rho_i^{\#S}(\mathbf{T}_S) \ge \#S$.*
(d) $\rho^{\#S}(\cdot) : \mathbb{R}_+^S \to \Delta^\circ(S)$ *is surjective.*[4]

Proof. Parts (a) and (b) are obvious from the definition of $\rho^k(\mathbf{T}_S)$. For the proofs of (c) and (d), see Appendix that is available on the JEBO website. □

3.2. Global accessibility

This subsection proves Theorem 1(a) through Lemmas 1 and 2. Lemma 2 gives a sufficient condition for global accessibility in terms of the function ρ^n, utilizing the fixed point theorem.

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[3] For any finite set X, $\#X$ denotes the cardinality of X.
[4] $\Delta^\circ(X) = \{f \in \Delta(X) : \mathrm{supp}(f) = X\}$ is the interior of $\Delta(X)$.

The basic idea of Lemma 2 is as follows. Suppose that a^* is $\mathbf{p}$-dominant, and if everyone in population i takes a_i^* after T_i,then the effectively discounted average action profile puts the probability of at least p_i on a_{-i}^* for each i. Then by the fixed point theorem there exists a perfect foresight path, for any initial state, in which everyone in population i takes a_i^* after T_i.

Lemma 2. *Let a^* be a $\mathbf{p}$-dominant equilibrium of G. If there exists $\mathbf{T} \in \mathbb{R}_+^n$ such that $\rho_i^n(\mathbf{T}) > p_i$ for every $i \in N$, then, for any revision speeds λ, there exists $\delta^* > 0$ such that $[a^*]$ is globally accessible under subjective discount rates δ with $-\lambda_i < \delta_i < \delta^*$ for every $i \in N$.*

Proof. Fix any state $x \in \prod_{i=1}^n \Delta(A_i)$. Define $\Phi(x, \mathbf{T})$ as the set of feasible paths from x such that for any $i \in N$, all the agents in population i choose action a_i^* if they get revision opportunities after time T_i; that is,

$$\Phi(x, \mathbf{T}) := \{\phi \in \Phi(x) : \phi \text{ is induced by some } \alpha \text{ with } \alpha_i(t) = [a_i^*] \text{ for } i \in N \text{ and } t \geq T_i\},$$

which is a nonempty, convex, and compact subset of $\Phi(x)$.

For any i, any $s \geq 0$, and any $\phi \in \Phi(x, \mathbf{T})$, we have $\phi_j(s)(a_j^*) \geq \max(0, 1 - \mathrm{e}^{\lambda_j(-s+T_j)})$ for any $j \neq i$. Therefore, we have

$$\prod_{j \neq i} \phi_j(s)(a_j^*) \geq \prod_{j \neq i} \max(0, 1 - \mathrm{e}^{\lambda_j(-s+T_j)}).$$

Let $\varepsilon = \min_{i \in N}(\rho_i^n(\mathbf{T}) - p_i) > 0$. Then there exists $\delta^* > 0$ such that

$$(\lambda_i + \delta_i) \int_t^\infty \mathrm{e}^{(\lambda_i+\delta_i)(t-s)} \left(\prod_{j \neq i} \max(0, 1 - \mathrm{e}^{\lambda_j(-s+T_j)}) \right) \mathrm{d}s \geq \rho_i^n(t, \mathbf{T}_{-i}) - \varepsilon$$

if $-\lambda_i < \delta_i < \delta^*$. Then, for any $t \geq T_i$, we have

$$\pi_{-i}(t, \phi_{-i})(a_{-i}^*) = (\lambda_i + \delta_i) \int_t^\infty \mathrm{e}^{(\lambda_i+\delta_i)(t-s)} \left(\prod_{j \neq i} \phi_j(s)(a_j^*) \right) \mathrm{d}s \geq (\lambda_i + \delta_i) \int_t^\infty \mathrm{e}^{(\lambda_i+\delta_i)(t-s)}$$
$$\times \left(\prod_{j \neq i} \max(0, 1 - \mathrm{e}^{\lambda_j(-s+T_j)}) \right) \mathrm{d}s \geq \rho_i^n(t, \mathbf{T}_{-i}) - \varepsilon \geq \rho_i^n(\mathbf{T}) - \varepsilon \geq p_i,$$

where the third inequality comes from Lemma 1(b) and $t \geq T_i$.

For any i and any $t \geq T_i$, since a^* is a $\mathbf{p}$-dominant equilibrium, $\pi_{-i}(t, \phi_{-i})(a_{-i}^*) \geq p_i$ implies that a_i^* is one of player i's best responses to $\pi_{-i}(t, \phi_{-i})$, so

$$\tilde{\beta}(x, \mathbf{T}, \phi) := \beta(x, \phi) \cap \Phi(x, \mathbf{T})$$

is nonempty. By applying the fixed point theorem to the restricted correspondence $\tilde{\beta}(x, \mathbf{T}, \cdot)$ from $\Phi(x, \mathbf{T})$ to itself, we obtain a fixed point $\phi^* \in \Phi(x, \mathbf{T})$. ϕ^* is a perfect foresight path from x to $[a^*]$. □

Now we show Theorem 1(a) using Lemmas 1 and 2.

Proof of Theorem 1(a). Since $\sum_i p_i < 1$, by Lemma 1(d), there exists $\mathbf{T}$ such that $\rho_i^n(\mathbf{T}) > p_i$ for every $i \in N$. Therefore, by Lemma 2, there exists $\delta^* > 0$ such that $[a^*]$ is globally accessible under δ with $-\lambda_i < \delta_i < \delta^*$. □

3.3. Absorption

This subsection proves Theorem 1(b) through Lemmas 1 and 3 in a parallel way to the previous subsection. Now we use function ρ^1. Let $\mathbf{p}_S = (p_i)_{i \in S}$.

The intuition of Lemma 3 is the following. Suppose that a^* is strict $\mathbf{p}$-dominant and it is not linearly absorbing (that is, for some nonempty subset S of N, agents in population $i \in S$ take $a_i \neq a_i^*$ at some period of time $T_i < \infty$). Then the discounted average action profile at T_i puts less than p_i at a_{-i}^*.

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Lemma 3. *Let a^* be a strict* **p**-*dominant equilibrium of G. If, for any positive integer m, there exists $\delta^m = (\delta_1^m, \ldots, \delta_n^m)$ such that $\delta_i^m > -1/m$ for every $i \in N$ and $[a^*]$ is not linearly absorbing under δ^m, then, for any $\varepsilon > 0$, there exists a nonempty subset S of N and $\mathbf{T}_S \in \mathbb{R}_+^S$ such that $(1-\varepsilon)\rho^1(\mathbf{T}_S) \leq \mathbf{p}_S$.*

Proof. Fix any $\varepsilon > 0$. Let ϕ^m be a perfect foresight path under δ^m induced by α^m from x^m such that $x^m(a^*) > 1 - \varepsilon$ and $\alpha^m(t) \neq [a^*]$ for some $t \geq 0$. Let $S^m \neq \emptyset$ be the set of i such that $\alpha_i^m(t) \neq [a_i^*]$ for some $t \geq 0$. Let $\mathbf{T}^m = (T_1^m, \ldots, T_n^m)$ be given by

$$T_i^m := \begin{cases} \inf\{t \in \mathbb{R}_+ : \alpha_i^m(t) \neq [a_i^*]\} & (i \in S^m), \\ \infty & (i \notin S^m). \end{cases}$$

Without loss of generality, we can assume $\min_{i \in N} T_i^m = 0$. (Otherwise, we can shift ϕ^m by $\min_i T_i^m$ to construct a perfect foresight path that starts at a state even closer to $[a^*]$.) Let $\mathbf{T} = (T_1, \ldots, T_n)$ be an accumulation point of $\{\mathbf{T}^m\}$, and let $S := \{i \in N : T_i < \infty\}$. Since $\min_i T_i^m = 0$ for all m, we have $\min_i T_i = 0$ and hence $S \neq \emptyset$. Then we will show that $\mathbf{T}_S$ satisfies the desired inequality.

Taking a subsequence if necessary, we have $S \subseteq S^m$ for any m and $T_i^m \to T_i$ as $m \to \infty$ for any $i \in S$. Fix any $i \in S$. For any $j \in S^m \setminus \{i\}$, by the definition of T_j^m, we have $\alpha_j^m(t)(a_j^*) = 1$ for $t < T_j^m$ and $\alpha_j^m(t)(a_j^*) \geq 0$ for $t \geq T_j^m$. Therefore,

$$\phi_j^m(t)(a_j^*) \geq x_j^m(a_j^*) \min(1, \mathrm{e}^{\lambda_j(T_j^m - t)}) \tag{1}$$

for any $t \geq 0$. For any $j \in N \setminus S^m$, we have

$$\phi_j^m(t)(a_j^*) \geq x_j^m(a_j^*) \tag{2}$$

for any $t \geq 0$. It follows from (1) and (2) that

$$\prod_{j \neq i} \phi_j^m(t)(a_j^*) \geq x_{-i}^m(a_{-i}^*) \prod_{j \in S^m \setminus \{i\}} \min(1, \mathrm{e}^{\lambda_j(T_j^m - t)}). \tag{3}$$

By the definition of T_i^m and the continuity of $\pi_{-i}^m(\cdot, \phi_{-i}^m)$, a_i^* is not i's unique best response to $\pi_{-i}^m(T_i^m, \phi_{-i}^m)$. (Here we add superscript m to π_{-i} since the "discounted" probability depends on subjective discount rates δ^m.) Since a^* is a strict **p**-dominant equilibrium, $\pi_{-i}^m(T_i^m, \phi_{-i}^m)(a_{-i}^*) \leq p_i$. By (3), we have

$$\begin{aligned}
&(1-\varepsilon)\rho_i^1(\mathbf{T}_S) < x_{-i}^m(a_{-i}^*)\rho_i^1(\mathbf{T}_S) \\
&= \lambda_i \int_{T_i}^{\infty} \mathrm{e}^{\lambda_i(T_i - t)} \left(x_{-i}^m(a_{-i}^*) \prod_{j \in S \setminus \{i\}} \min(1, \mathrm{e}^{\lambda_j(T_j - t)}) \right) \mathrm{d}t \\
&\leq \operatorname{liminf}_{m \to \infty} (\lambda_i + \delta_i^m) \int_{T_i^m}^{\infty} \mathrm{e}^{(\lambda_i + \delta_i^m)(T_i^m - t)} \left(x_{-i}^m(a_{-i}^*) \prod_{j \in S^m \setminus \{i\}} \min(1, \mathrm{e}^{\lambda_j(T_j^m - t)}) \right) \mathrm{d}t \\
&\leq \operatorname{liminf}_{m \to \infty} (\lambda_i + \delta_i^m) \int_{T_i^m}^{\infty} \mathrm{e}^{(\lambda_i + \delta_i^m)(T_i^m - t)} \left(\prod_{j \neq i} \phi_j^m(t)(a_j^*) \right) \mathrm{d}t = \operatorname{liminf}_{m \to \infty} \pi_{-i}^m(T_i^m, \phi_{-i}^m)(a_{-i}^*) \leq p_i
\end{aligned}$$

for any $i \in S$. □

Now we prove Theorem 1(b) using Lemmas 1 and 3.

Proof of Theorem 1(b). Suppose that, for any positive integer m, there exists $\delta^m = (\delta_1^m, \ldots, \delta_n^m)$ such that $\delta_i^m > -1/m$ for every $i \in N$ and $[a^*]$ is not linearly absorbing under δ^m. Then, by Lemma 3, for any $\varepsilon > 0$, there exist S and $\mathbf{T}_S \in \mathbb{R}_+^S$ such that $(1-\varepsilon)\rho^1(\mathbf{T}_S) \leq \mathbf{p}_S$. By Lemma 1(a):

$$1 - \varepsilon = (1-\varepsilon)\sum_{i \in S} \rho_i^1(\mathbf{T}_S) \leq \sum_{i \in S} p_i \leq \sum_{i=1}^{n} p_i.$$

Therefore, $\sum_{i=1}^{n} p_i \geq 1$. □

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4. Extensions and other applications

4.1. Extensions

4.1.1. Rationalizable foresight dynamics

We consider rationalizable foresight dynamics introduced by Matsui and Oyama, who apply the concept of rationalizability by Bernheim (1984) and Pearce (1984) to large population dynamics. We relax the assumption of consistency of expectation while retaining that of common knowledge of rationality. Every agent in the society maximizes the expected payoff based on her belief on the feasible path in the future. Unlike those in perfect foresight dynamics, expected paths do not have to be correct, as long as they are consistent with common knowledge of rationality.

Let $\Phi^0(x)$ denote the set of all feasible paths, and for $k \geq 1$ define Φ^k by

$$\Phi^k = \left\{ \phi \in \Phi^{k-1} : \forall i \in N, \forall t \in \mathbb{R}_+, \mathrm{supp}(\alpha_i(t)) \subseteq \bigcup_{\psi \in \Phi^{k-1}, \phi(t)=\psi(t)} \mathrm{br}(\pi_{-i}(t, \psi_{-i})) \right\}.$$

The set $\Phi^* = \bigcap_{k=0}^{\infty} \Phi^k$ is called the set of *rationalizable foresight paths*. It is easy to see that any perfect foresight path is a rationalizable foresight path (Matsui and Oyama, Claim 4.1). Matsui and Oyama (Proposition 3.2) also show that a feasible path ϕ with initial state x is a rationalizable foresight path if and only if $\phi \in \beta(x, \psi)$ for some $\psi \in \Phi^*$.

$x^* \in \prod_{i=1}^{n} \Delta(A_i)$ is *globally accessible under rationalizable foresight* if there exists a rationalizable foresight path from any state that converges to x^*. x^* is *absorbing under rationalizable foresight* if there exists an open neighborhood U of x^* such that any rationalizable foresight path from any point of U converges to x^*. $x^* = [a^*]$ is *linearly absorbing under rationalizable foresight* if there exists an open neighborhood U of $[a^*]$ such that for any $x \in U$, the path induced by $\alpha(t) = [a^*]$ for all $t \geq 0$ is a unique rationalizable foresight path from x. Note that (linear) absorption under rationalizable foresight is a stronger concept than (linear) absorption under perfect foresight. Matsui and Oyama (Example 3.1) give an example of 2 × 2 game, in which a state is absorbing under perfect foresight but not under rationalizable foresight.

Proposition 1. *Let $a^* \in A$.*

(a) *If a^* is a $\mathbf{p}$-dominant equilibrium of G with $\sum_i p_i < 1$, then, for any revision speeds λ, there exists $\delta^* > 0$ such that $[a^*]$ is globally accessible under rationalizable foresight and subjective discount rates δ with $-\lambda_i < \delta_i < \delta^*$ for every $i \in N$.*
(b) *If a^* is a strict $\mathbf{p}$-dominant equilibrium of G with $\sum_i p_i < 1$, then, for any revision speeds λ, there exists $\delta_* < 0$ such that $[a^*]$ is linearly absorbing under rationalizable foresight and subjective discount rates δ with $\delta_i > \delta_*$ for every $i \in N$.*

Part (a) is obvious from the observation that any perfect foresight path is a rationalizable foresight path. To prove (b), we show the following lemma, which is an extension of Lemma 3.

Lemma 4. *Let a^* be a strict $\mathbf{p}$-dominant equilibrium of G. If, for any positive integer m, there exists $\delta^m = (\delta_1^m, \ldots, \delta_n^m)$ such that $\delta_i^m > -1/m$ for every $i \in N$ and $[a^*]$ is not linearly absorbing under rationalizable foresight and δ^m, then, for any $\varepsilon > 0$, there exists a nonempty subset S of N and $\mathbf{T}_S \in \mathbb{R}_+^S$ such that $(1-\varepsilon)\rho^1(\mathbf{T}_S) \leq \mathbf{p}_S$.*

Proof. Fix any $\varepsilon > 0$. Then there exists a rationalizable foresight path under δ^m induced by α^m from x^m to a^* such that $x^m(a^*) > 1 - \varepsilon$ and $\alpha^m(t) \neq [a^*]$ for some $t \geq 0$. Let $\mathbf{T}^m = (T_1^m, \ldots, T_n^m)$ be given by

$$T_i^m := \inf_{\phi \in \Phi^m(x^m)} \inf\{t \in \mathbb{R}_+ : \phi \text{ is induced by } \alpha, \alpha_i(t) \neq [a_i^*]\},$$

where $\Phi^m(x^m)$ is the set of rationalizable foresight paths from x^m under δ^m. Then we can show that there exists $\mathbf{T}_S$ that satisfies the desired inequality as in Lemma 3. □

This lemma and Lemma 1(a) proves Part (b) of Proposition 1.

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4.1.2. **p**-Best Response Sets

We consider a set-valued extension of **p**-dominance due to Tercieux.

Definition 4. Let $B_i \subseteq A_i$, $B = \prod_{i=1}^n B_i$, and $B_{-i} = \prod_{j \neq i} B_j$.

(a) B is a ***p**-best response set* if, for any $i \in N$ and any $x_{-i} \in \Delta(A_{-i})$, $x_{-i}(B_{-i}) \geq p_i$ implies $\mathrm{br}_i(x_{-i}) \cap B_i \neq \emptyset$.[5]
(b) B is a *strict* **p**-*best response set* if, for any $i \in N$ and any $x_{-i} \in \Delta(A_{-i})$, $x_{-i}(B_{-i}) > p_i$ implies $\mathrm{br}_i(x_{-i}) \subseteq B_i$.

p-Best response set is a set-valued extension of the concept of (strict) **p**-dominance. $\{a^*\}$ is a (strict) **p**-best response set if and only if a^* is a (strict) **p**-dominant equilibrium. We say that B is a minimal (strict) **p**-best response set if it is a (strict) **p**-best response set and no proper subset of B is a (strict) **p**-best response set. There always exist (strict) minimal **p**-best response sets since A is a strict **p**-best response set for any $\mathbf{p} \in [0, 1]^n$ and G is a finite game, in contrast to (strict) **p**-dominant equilibria that may fail to exist.

Next we define the set-valued stability concepts in perfect foresight dynamics.

Definition 5. Let $x \in \prod_{i=1}^n \Delta(A_i)$ and $Y \subseteq \prod_{i=1}^n \Delta(A_i)$.

(a) Y is a *globally accessible set* if, for any $x \in \prod_{i=1}^n \Delta(A_i)$, there exists a perfect foresight path that converges to Y.[6]
(b) Y is an *absorbing set* if there exists a neighborhood U of Y such that for any perfect foresight path from any point of U converges to Y.
(c) $Y = \prod_{i=1}^n \Delta(B_i)$ for some $B_i \subseteq A_i$ is a *linearly absorbing set* if there exists a neighborhood U of Y such that for any perfect foresight path from any point of U is induced by α with $\alpha_i(t)(B_i) = 1$ for any $i \in N$ and $t \geq 0$.

Note that Oyama (2002) and Tercieux (2006) give different definitions to “globally accessible sets” and “absorbing sets”, respectively.

It is easy to see that $\{x^*\}$ is a globally accessible (absorbing) set if and only if x^* is a globally accessible (absorbing) state, and $\{[a^*]\}$ is a linearly absorbing set if and only if $[a^*]$ is linearly absorbing. Thus the set-valued concepts are extensions of global accessibility and (linear) absorption in the original sense.

Below is a set-valued extension of Theorem 1. The proof is analogous to that of Theorem 1.

Proposition 2. *Let $B_i \in A_i$ and $B = \prod_{i=1}^n B_i$.*

(a) *If B is a* **p**-*best response set of G with $\sum_{i=1}^n p_i < 1$, then, for any revision speeds λ, there exists $\delta^* > 0$ such that $\prod_{i=1}^n \Delta(B_i)$ is globally accessible under subjective discount rates δ with $-\lambda_i < \delta_i < \delta^*$ for every $i \in N$.*
(b) *If B is a strict* **p**-*best response set of G with $\sum_{i=1}^n p_i < 1$, then, for any revision speeds λ, there exists $\delta_* < 0$ such that $\prod_{i=1}^n \Delta(B_i)$ is linearly absorbing under subjective discount rates δ with $\delta_i > \delta_*$ for every $i \in N$.*

This section presents extensions with respect to different perturbations separately. Here we point out that these extensions can be presented in a single model. That is, we can show global accessibility and absorption of **p**-dominant sets under rationalizable foresight.

4.2. Other applications

In this subsection we assume that revision speeds are homogeneous among populations. We set $\lambda_i = 1$ for every $i \in N$ without loss of generality.

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[5] For any $Y \subseteq X$ and $f \in \Delta(X)$, we define $f(Y) = \sum_{y \in Y} f(y)$.

[6] Let $d(x, y)$ denote the distance between x and y. $d(x, Y) = \inf_{y \in Y} d(x, y)$. ϕ converges to Y if $\lim_{t \to \infty} d(\phi(t), Y) = 0$.

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4.2.1. *u*-Dominant equilibria

We define the following concept, which is a slight modification of u-dominance defined by Kojima.

Definition 6. Let $a^* \in A$. a^* is a *u-dominant equilibrium* of G if for any $i \in N$ and $x_{-i} \in \Delta(A_{-i})$, $x_{-i}(\{a_{-i} : \#\{j \neq i : a_j = a_j^*\} \leq k-1\}) \leq k/n$ for every $k \in \{1, \dots, n-1\}$ implies $\{a_i^*\} = \mathrm{br}_i(x_{-i})$.

The concept of u-dominance is an extension of $\mathbf{p}$-dominance in two-player games. In two-player games, a^* is a u-dominant equilibrium if and only if it is strict (p, p)-dominant with $p < 1/2$. Following Oyama, we say that $[a^*]$ is *linearly stable* if, from any state, the path induced by $\alpha(t) = [a^*]$ is a perfect foresight path. Clearly, if $[a^*]$ is linearly stable, then it is globally accessible.

Now we show a slight variant of Kojima's result.[7]

Proposition 3. *Let $a^* \in A$.*

(a) *If a^* is u-dominant, then there exists $\delta^* > 0$ such that $[a^*]$ is linearly stable under subjective discount rates δ with $-1 < \delta_i < \delta^*$ for every $i \in N$.*
(b) *If a^* is u-dominant, then there exists $\delta_* < 0$ such that $[a^*]$ is linearly absorbing under subjective discount rates δ with $\delta_i > \delta_*$ for every $i \in N$.*

Proof. Let $\mathbf{0} = (0)_{i \in N}$. First observe that $\rho_i^k(\mathbf{0}) = 1/n$ for any $k \in \{1, \dots, n\}$ since $\lambda_i = 1$ for any $i \in N$ and, by continuity of utility functions u, there exists $\varepsilon > 0$ such that for any $i \in N$ and $x_{-i} \in \Delta(A_{-i})$, $x_{-i}(\{a_{-i} : \#\{j \neq i : a_j = a_j^*\} \leq k-1\}) \leq k/n + \varepsilon$ for every $k \in \{1, \dots, n-1\}$ implies $\{a_i^*\} = \mathrm{br}_i(x_{-i})$.

(a) For any $x \in \prod_{i=1}^n \Delta(A_i)$, let ϕ be a feasible path induced by α such that $\alpha(t) = a^*$ for any $t \geq 0$. By construction, for any $\varepsilon > 0$, there exists $\delta^* > 0$ such that, for any δ with $-1 < \delta_i < \delta^*$ for every $i \in N$, $\pi_{-i}(t, \phi)$ satisfies $\sum_{a_{-i}:\#\{j \neq i : a_j = a_j^*\} \leq k-1} \pi_{-i}(t, \phi)(a_{-i}) \leq \sum_{l=1}^k \rho_i^l(\mathbf{0}) + \varepsilon = k/n + \varepsilon$ for every $i \in N$, $t \geq 0$ and $k \in \{1, \dots, n\}$. Since $\varepsilon > 0$ is arbitrary and a^* is a u-dominant equilibrium, a_i^* is a unique best response at every moment t and ϕ is a perfect foresight path. Since δ^* can be chosen independently of the initial state x, $[a^*]$ is linearly stable.
(b) For any $\varepsilon > 0$, there exists $\delta_* < 0$ such that, for any δ with $\delta_i > \delta_*$ for every $i \in N$, there exists a neighborhood U of $[a^*]$ such that, for every feasible path ϕ from U, $\sum_{a_{-i}:\#\{j \neq i : a_j = a_j^*\} \leq k-1} \pi_{-i}(0, \phi)(a_{-i}) \leq \sum_{l=n-k+1}^n \rho_i^l(\mathbf{0}) + \varepsilon = k/n + \varepsilon$. Since $\varepsilon > 0$ is arbitrary and a^* is u-dominant, a_i^* is a unique best response of i against $\pi(0, \phi)$, showing linear absorption. $\square$

4.2.2. Unanimity games

This subsection investigates perfect foresight dynamics on a unanimity game.

G is called a *unanimity game* if $A_i = \{0, 1\}$ for any $i \in N$, and

$$u_i(a) = \begin{cases} y_i & (a = \mathbf{0}), \\ z_i & (a = \mathbf{1}), \\ 0 & (\text{otherwise}) \end{cases}$$

for some $y_i, z_i > 0$, where $\mathbf{0}{:=}(0, \dots, 0)$ and $\mathbf{1}{:=}(1, \dots, 1)$. In a unanimity game, players get positive payoffs if and only if all of them choose the same action.

Lemma 5 (OTH, Proposition 5.2.2). *$[\mathbf{1}]$ is linearly absorbing under any positive subjective discount rates δ if and only if, for any $\mathbf{T} \in \mathbb{R}_+^n$, there exists i such that $z_i \rho_i^1(\mathbf{T}) \geq y_i \rho_i^n(\mathbf{T})$.*

Proposition 4. *If $\sum_{i=1}^n y_i/z_i \leq n$, then $[\mathbf{1}]$ is linearly absorbing under any positive subjective discount rates δ.*

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[7] The main difference between our paper and Kojima is that he restricts his attention to a class of symmetric games which he calls PIM games.

Proof. Since $\sum_{i=1}^{n} y_i/z_i \leq n$, we have

$$\sum_{i=1}^{n} \frac{y_i}{z_i} \leq n \leq \sum_{i=1}^{n} \frac{\rho_i^1(\mathbf{T})}{\rho_i^n(\mathbf{T})}$$

for any $\mathbf{T} \in \mathbb{R}_+^n$ by Lemma 1(c) and the assumption that $\lambda_i = 1$ for every $i \in N$. Therefore, there exists i such that $y_i/z_i \leq \rho_i^1(\mathbf{T})/\rho_i^n(\mathbf{T})$. Then it follows from Lemma 5 that $[\mathbf{1}]$ is absorbing. □

We end this section with the following conjecture. The *Nash products* of $\mathbf{0}$ and $\mathbf{1}$ are given by $\prod_{i=1}^{n} y_i$ and $\prod_{i=1}^{n} z_i$, respectively.

Conjecture. *If $\mathbf{1}$'s Nash product is higher than or equal to $\mathbf{0}$'s, then there exists $\delta^* > 0$ such that $[\mathbf{1}]$ is globally accessible under subjective discount rates δ with $-1 < \delta_i < \delta^*$ for every $i \in N$.*

This conjecture is true for $n = 2$, which is the case that Matsui and Matsuyama investigate. OTH also prove for the case of $n = 3$ where two of the three players have the same value of z_i/y_i.

Hofbauer (1999) investigates unanimity games in his spatio-temporal game approach and selects the equilibrium with a higher Nash product.

Note that Propositions 3 and 4 rely explicitly on homogeneous revision speeds. Also, Kim (1996) and Kojima (2006) essentially rely on it, assuming a single population setting.

5. Conclusion

This paper analyzed perfect foresight dynamics on asymmetric games. If the base game has a **p**-dominant equilibrium a^* with $\sum_i p_i < 1$, then $[a^*]$ is globally accessible. Moreover, if a^* is strict **p**-dominant with $\sum_i p_i < 1$, then $[a^*]$ is linearly absorbing. This paper also investigated robustness and extensions of our result with respect to rationalizable foresight and set-valued concepts. Finally we applied our method to games with u-dominant equilibria and unanimity games.

Results in this paper suggest a coincidence among perfect foresight dynamics and other equilibrium selections, especially robustness to incomplete information and the global game approaches. Actually, Kajii and Morris show that if the complete information game has a **p**-dominant equilibrium a^* with $\sum_i p_i < 1$, then a^* is robust to incomplete information; Frankel et al. (2003) show that if the complete information game has a **p**-dominant equilibrium a^* with $\sum_i p_i < 1$, then a^* is selected in global games due to Carlsson and van Damme (1993) independently of the noise structure. We need future research to reveal what causes this coincidence.

Acknowledgement

We are grateful to the Associate Editor and anonymous referees for comments.

Appendix A. Supplementary Data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.jebo.2007.07.002.

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Please cite this article in press as: Kojima, F., Takahashi, S., **p**-Dominance and perfect foresight dynamics, Journal of Economic Behavior and Organization (2007), doi:10.1016/j.jebo.2007.07.002