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Mixed strategies in games of capacity manipulation in hospital–intern markets

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Abstract We investigate games of capacity manipulation in hospital-intern markets as proposed by Konishi and Ünver (Soc Choice Welfare, in press). While Konishi and Ünver (Soc Choice Welfare, in press) show that there may not exist a pure-strategy Nash equilibrium in general, there exists a mixed-strategy Nash equilibrium in such a game. We show that every hospital weakly prefers a Nash equilibrium to any "larger" capacity profiles, whether the equilibrium is in pure or mixed strategies. In particular, a Nash equilibrium is weakly preferred by hospitals to the outcome that results from truthful reporting.

1 Introduction

Recently much interest has been growing on manipulation and implementation in *two-sided matching markets*.[1] The literature of two-sided matching is widely applied to various entry-level labor markets, such as medical residency match in the US and the UK (Roth 1984a, 1991; Roth and Peranson 1999). Consider a hospital–intern market, in which a hospital is matched to interns at most up to its capacity and each intern is matched to at most one hospital. A property called stability has been shown to be important. A matching is said to be stable if (1) no intern or hospital prefer being unmatched to being matched to the assigned partner, and (2) no pair of a hospital and an intern who are unmatched to each other prefer each other to the assigned partners. A matching mechanism is stable if it always produces a stable matching with respect to reported preferences and capacities.

Given that stability plays a crucial role in real-life applications, it is important to investigate strategic behavior under stable mechanisms. A classical impossibility

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[1] See Roth and Sotomayor (1990) for a comprehensive survey.

theorem of Roth (1984b) shows that no stable matching mechanism is immune to manipulation via preference misrepresentation. More recently, Sönmez (1997) shows that no stable matching mechanism is immune to manipulation via capacities: That is, for any stable matching mechanism, a hospital sometimes benefits by reporting a smaller number of needed positions than it really has, thereby being matched with (a smaller number of) interns who it prefers to those whom it would have been matched with.[2] Konishi and Ünver (2005) investigate the problem of capacity manipulation further by equilibrium analysis. Their main results are (1) a game of capacity manipulations may not have a pure-strategy Nash equilibrium, (2) if there exists a pure-strategy Nash equilibrium, then every hospital weakly prefers the equilibrium outcome to the outcome generated by truthful reporting, while every intern weakly prefers the latter to the former.

This note complements the analysis of Konishi and Ünver (2005) by investigating mixed strategies in capacity-reporting games. While Konishi and Ünver (2005) show that there may not exist a pure-strategy Nash equilibrium in general, we note that there exists a mixed-strategy Nash equilibrium in such a game. We show that the welfare properties carry over to mixed-strategy Nash equilibria. More specifically, every hospital weakly prefers a Nash equilibrium to any "larger" capacity profiles (in the sense of first-order stochastic dominance), whether the equilibrium is in pure or mixed strategies. In particular, a Nash equilibrium is weakly preferred by every hospital to the outcome obtained by truthful reporting.[3]

2 Model and result

Formally, a hospital–intern market is given by quadruple (H, I, q, R) where

1. $H = \{h_1, \dots, h_m\}$ is a set of hospitals.
2. $I = \{i_1, \dots, i_n\}$ is a set of interns.
3. $q = (q_{h_1}, \dots, q_{h_m})$ is a profile of hospital capacities, where q_h is the capacity of hospital h.
4. $R = (R_{h_1}, \dots, R_{h_m}, R_{i_1}, \dots, R_{i_n})$ is a profile of preferences, where R_h is the preferences of hospital h and R_i is the preferences of intern i (associated strict preferences are denoted by P_h and P_i, respectively.)

While interns have preferences on hospitals and being unmatched, $X^H \equiv H \cup \{\emptyset\}$, hospitals have preferences on the set of groups of interns $X^I \equiv 2^I$. Preference relation R_h of hospital h is *responsive* (Roth 1985) if we have (1) for any $i, j \in I$ and any $J \subseteq I \backslash \{i, j\}$, we have $(J \cup \{i\}) R_h (J \cup \{j\}) \iff \{i\} R_h \{j\}$, and (2) for any $i \in I$ and $J \subseteq I \backslash \{i\}$, we have $(J \cup \{i\}) R_h J \iff \{i\} R_h \emptyset$. Throughout the paper, we assume that each hospital has responsive preferences.

Let $N \equiv H \cup I$. A matching μ is a function from N to $X^I \cup X^H$ such that $\mu_i \in X^H$, $|\mu_h| \leq q_h$ and $\mu_i = h$ if and only if $i \in \mu_h$ for any $i \in I$ and $h \in H$.[4] A matching is *stable* if (1) there exists no hospital h and intern i such that for some

[2] Sönmez (1999) investigates closely related manipulations via pre-arrangement, that is, manipulations by arranging contracts privately before the centralized match is run.

[3] We note that capacity manipulations are a real concern rather than just theoretical possibility. For instance, Abdulkadiroğlu et al. (2005) report that many high schools in New York City withheld their quotas in order to admit students they preferred.

[4] For $v \in N$, μ_v denotes the value of function μ evaluated at v.

$J \subseteq \mu_h$ with $|J| < q_h$, we have $(J \cup \{i\}) P_h \mu_h$ and $h P_i \mu_i$, and (2) there exists no agent $v \in N$ such that for some agent $y \in \mu_v$ we have $\emptyset P_v y$.

We fix H, I and R throughout the paper. Gale and Shapley (1962) show that there exists a stable matching and that there are stable matchings that are the most preferred by all hospitals (the hospital-optimal stable match) and by all interns (the intern-optimal stable match), respectively. For each capacity profile q, we denote by $\varphi^I(q)$ and $\varphi^H(q)$ the intern-optimal stable matching and the hospital-optimal stable matching of (H, I, q, R), respectively. For $V \in \{I, H\}$, $\varphi_h^V(q)$ denotes interns who are matched to $h \in H$ in problem (H, I, q, R) under matching rule φ^V.

Let $V \in \{I, H\}$. A *capacity-reporting game* under φ^V is a normalform game in which (1) the set of players is H, (2) each hospital h reports $\tilde{q}_h \in \{0, \ldots, q_h\}$, and (3) the outcome is given by the stable matching $\varphi^V(\tilde{q})$ with respect to reported capacities $\tilde{q} = (\tilde{q}_h)_{h \in H}$.[5] Note that interns are passive players in this game. Unlike Konishi and Ünver (2005), we allow for mixed strategies in the capacity manipulation game. A mixed strategy of h is a probability distribution on capacities. Formally, a mixed strategy of h is $p_h = (p_h^0, p_h^1, \ldots, p_h^{q_h}) \in \Delta_h \equiv \{(p_h^0, p_h^1, \ldots, p_h^{q_h}) \in \mathbb{R}^{q_h}; p_h^t \geq 0, \sum_t p_h^t = 1\}$, where p_h^t is the probability that h reports that its capacity is t. We assume that preferences of each agent $v \in N$ satisfy the von-Neumann Morgenstern expected utility hypothesis.[6] We extend notation as follows; $\varphi^V(p)$ denotes a distribution on matchings induced by mixed strategy p under mechanism φ^V; R_v denotes agent v's preference over distributions on matchings as well as non-stochastic matchings. While Konishi and Ünver (2005) show by examples that there may not be any pure-strategy Nash equilibrium in capacity-reporting games, it is straightforward to see that there exists a (possibly) mixed Nash equilibrium.

Proposition 1 *Let $V \in \{I, H\}$. For any problem (H, I, q, R), there exists a (possibly mixed) Nash equilibrium in a capacity-reporting game under φ^V.*

Proof Immediate from Nash (1951). □

Next we consider a welfare property of Nash equilibria. A mixed-strategy profile p is *first-order stochastically dominated* by another profile $\tilde{p}$ (denoted $p \preceq \tilde{p}$) if

$$\sum_{s \geq t} p_h^s \leq \sum_{s \geq t} \tilde{p}_h^s \quad \text{for any } h \in H \text{ and } t \in \{0, 1, \ldots, q_h\}.$$

The following result is taken from Konishi and Ünver (2005).

Result 1 (Capacity Lemma) *Let $V \in \{I, H\}$. In capacity-reporting games under φ^V, a hospital's capacity underreport in pure strategies makes all other hospitals weakly better off and all interns weakly worse off.*

Now we state the main result concerning the welfare of hospitals and interns.

[5] We assume that hospitals cannot overreport their positions. Note that if one assumes that $|J| > q_h$ implies $\emptyset P_h J$ for any $h \in H$ and $J \in X^I$ (as usually assumed in the literature), then overreporting is a weakly dominated strategy.

[6] Let X be a finite set and Δ be the set of distributions over X. A preference relation R over Δ is said to satisfy the expected utility hypothesis if there exists a function $u : X \to \mathbb{R}$ such that, for any $p = (p_x)_{x \in X}$, $\tilde{p} = (\tilde{p}_x)_{x \in X} \in \Delta$, we have $p R \tilde{p}$ if and only if $\sum_{x \in X} p_x u(x) \geq \sum_{x \in X} \tilde{p}_x u(x)$.

Theorem 1 *Let $V \in \{H, I\}$. If p is a Nash equilibrium in the capacity-reporting game under φ^V and another profile $\tilde{p}$ satisfies $p \leq \tilde{p}$, then (i) every hospital weakly prefers $\varphi^V(p)$ to $\varphi^V(\tilde{p})$, and (ii) every intern weakly prefers $\varphi^V(\tilde{p})$ to $\varphi^V(p)$.*

Proof Suppose that p is an equilibrium of this game under φ^V and suppose $p \leq \tilde{p}$. Let $h \in H$ an arbitrary hospital. Since p is an equilibrium, $\varphi_h^V(p) R_h \varphi_h^V(\tilde{p}_h, p_{-h})$ follows. Since $p \leq \tilde{p}$, from a well-known inequality of first-order stochastic dominance and the Capacity Lemma, we have that $\varphi_h^V(\tilde{p}_h, p_{-h}) R_h \varphi_h^V(\tilde{p})$. By transitivity we have $\varphi_h^V(p) R_h \varphi_h^V(\tilde{p})$. On the other hand, a direct application of the Capacity Lemma and the first-order stochastic dominance implies $\varphi_i^V(p) R_i \varphi_i^V(\tilde{p})$ for each intern $i \in I$. □

Theorem 1 generalizes Theorem 3 of Konishi and Ünver (2005), which shows the same result in cases where pure-strategy Nash equilibria exist.

Note that we do not assume any specific functional form of the utility function as long as it satisfies expected utility hypothesis. In other words, Theorem 1 holds for any risk attitude of agents.

Finally we state a corollary of Theorem 1. On the one hand, every hospital weakly prefers a Nash equilibrium to the truthful capacity reporting. On the other hand, every intern weakly prefers the truthful capacity reporting to a Nash equilibrium.

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