FINDING ALL STABLE MATCHINGS WITH COUPLES

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Abstract. In two-sided matching markets in which some doctors form couples, we present an algorithm that finds all the stable matchings whenever one exists, and otherwise shows that there is no stable matching. Extending the methodology of Echenique and Yenmez (2006), we characterize the set of stable matchings as the fixed points of a monotone decreasing function with respect to a certain partial order. Based on that result, an algorithm is presented that finds all the stable matchings, if any, in a market with couples.

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1. Introduction

The theory of two-sided matching is applied to various labor markets such as the one for new doctors and hospitals in the United States (National Residency Matching Program).[1] A matching is stable if no individual agents are matched to unacceptable partners, and no groups of doctors and hospitals prefer matching among themselves to matching with their current partners. Stability has been regarded as an essential property for the success of the matching. However, a stable matching

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[1]See Roth and Sotomayor (1990) for a survey of the theory in the first three decades starting with Gale and Shapley (1962). The theory has recently been applied to assignment of students to schools as well: see for example Balinski and Sönmez (1999), Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005), Abdulkadiroğlu, Pathak, and Roth (2005) and Erdil and Ergin (2006).

may not exist and finding a stable matching is often difficult even when one exists, if some doctors form couples (Roth, 1984).[2]

We present an algorithm that finds all the stable matchings whenever one exists, and otherwise shows that there is no stable matching in a market with couples. We characterize the set of stable matchings as the fixed points of a certain function, and show that the function is monotone decreasing with respect to a certain partial order. Based on these results, an algorithm is available to find all the stable matchings, if any, in a market with couples.

Our approach is an extension of an elegant paper by Echenique and Yenmez (2006) to markets with couples. They consider many-to-one matching markets where doctors are single but have strict preferences over pairs of the hospitals they attend and their colleagues. They first characterize the set of stable matchings as the fixed points of a decreasing function. Based on such properties, they propose an algorithm to find all the stable matchings that is very fast in many instances.

An alternative model of matching markets with couples (Dutta and Masso, 1997) is studied by Echenique and Yenmez (2006). While close in spirit, the standard model of couples we analyze is logically unrelated to theirs. In a number of real markets such as NRMP, couples seem to have preferences similar to the one we consider; for example, NRMP allows couples to express such preferences. The current paper complements Echenique and Yenmez (2006) by providing an algorithm that is applicable to a number of labor markets.

While our proof is analogous to Echenique and Yenmez (2006), obtaining an algorithm in matching markets with couples is important at least for two reasons.[3] First, couples are present in several labor markets such as NRMP, and no nontrivial algorithm has been known to always find even *one* stable matching in the presence of couples even when one exists. Indeed, Roth and Peranson (1999) and Klaus, Klijn, and Masso (2007) note that the algorithm used in NRMP may fail to find a stable matching even if one exists.[4] The alternative algorithm proposed here may improve upon the existing ones, guaranteeing finding a stable matching whenever one exists.

Second, the new algorithm may be useful for selection over (possibly multiple) stable matchings. Roth and Peranson (1999) note that a crisis of confidence resulted in a change of the matching mechanism in the 1990s, as some medical students suspected, among other things,

[2]Klaus and Klijn (2005) consider conditions under which stable matchings exist in the presence of couples. Other papers studying matching markets with couples include Klaus, Klijn, and Nakamura (2006), Klaus and Klijn (2007) and Klaus, Klijn, and Masso (2007).

[3]Regarding the proof technique, new arguments are developed to cope with special nature of couples, especially in Lemma 1.

[4]Note that, however, no instance of failing to find a stable matching has been reported in NRMP (Roth and Peranson, 1999).

that the matching mechanism favored hospitals in expense of doctors. To investigate the issue, Roth and Peranson (1999) conduct a simulation over several matching algorithms and find little differences in the resulting stable matching when different algorithms are applied. However, the stable matchings found by their method may not exhaust the entire set of stable matchings.[5] Indeed, there may be a different stable matching in which welfare of participants is quite different from the ones found in the simulation. Since the new algorithm can find all the stable matchings, it can help the matching authority to select the outcome that is agreed to be the most preferred, among all the stable matchings.

2. Model and Result

A matching problem is a tuple $(H, D, (R_i)_{i\in H\cup D})$. H and D are the sets of hospitals and doctors, respectively. D is partitioned into S and C, where S is the set of single doctors and C is the set of couples. Each couple is denoted by $c = (m_c, f_c)$, where m_c and f_c denote male and female members of c, respectively. Let $M = \{m_c | c \in C\}$ and $F = \{f_c | c \in C\}$ be the sets of males and females that form couples. We write $N = D \cup H$. Denoting being unmatched by $\emptyset$, we define $\tilde{S} = S \cup \{\emptyset\}$, $\tilde{H} = H \cup \{\emptyset\}$ and $\tilde{D} = S \cup M \cup F \cup \{\emptyset\}$. Each $s \in S$ has a preference relation over $\tilde{H}$, each $c \in C$ over $\tilde{H}^2$ and each $h \in H$ over $\tilde{D}$.[6] Preference relation R_i of agent $i \in N$ is assumed to be strict. We denote the strict part of R_i by P_i. As a notational convention, we assume that $xR_\emptyset y$ for any x, y.

A **prematching** is a mapping ν defined on the set N, satisfying

$$\begin{aligned} &\nu_s \in \tilde{H} && \text{for any } s \in S, \\ &\nu_c \in \tilde{H}^2 && \text{for any } c \in C, \\ &\nu_h \in \tilde{D} && \text{for any } h \in H. \end{aligned}$$

We denote the set of all prematchings by Φ.

Definition 1. A prematching ν is a **matching** if (1) if $\nu_s = h \in H$, then $\nu_h = s$, (2) if $\nu_c = (h, h')$ and $h \in H$, then $\nu_h = m_c$, and if $\nu_c = (h, h')$ and $h' \in H$, then $\nu_{h'} = f_c$, and (3) if $\nu_h = s \in S$ then $\nu_s = h$, if $\nu_h = m_c \in M$ then $\nu_c = (h, h')$ with $h' \in \tilde{H}$, and if $\nu_h = f_c \in F$ then $\nu_c = (h', h)$ with $h' \in \tilde{H}$.

[5]In a simple market as assumed in most literature of two-sided matching (with no couples in particular), their simulations would have found the best and worst stable matchings for doctors and hospitals, respectively. In such cases, Immorlica and Mahdian (2005) and Kojima and Pathak (2006) give theoretical accounts for the observation of Roth and Peranson (1999) that most doctors are matched to the same hospitals in all the stable matchings when a large number of agents participate in the market. However, their methods do not bound stable matchings in a market with couples, which is the case in NRMP.

[6]We assume that each hospital hires at most one doctor for simplicity.

A matching μ is **individually rational** if $\mu_s R_s \emptyset$ for every $s \in S$; for every $c \in C$, letting $\mu_c = (h, h') \in \tilde{H}^2$ we have $\mu_c R_c (\emptyset, h')$, $\mu_c R_c (h, \emptyset)$ and $\mu_c R_c (\emptyset, \emptyset)$; and $\mu_h R_h \emptyset$ for any $h \in H$. A pair $(s, h) \in S \times H$ is a **block** of μ if $h P_s \mu_s$ and $s P_h \mu_h$. A tuple $(c, h, h') \in C \times \tilde{H}^2$ is a **block** of μ if $(h, h') P_c \mu_c$, $m_c R_h \mu_h$ and $f_c R_{h'} \mu_{h'}$. A matching is **stable** if it is individually rational and it is not blocked.

For any prematching ν, define **available sets of partners** as

$$\begin{aligned}
A_s(\nu) &= \{h \in \tilde{H} | s R_h \nu_h\}, \\
A_c(\nu) &= \{(h, h') \in \tilde{H}^2 | m_c R_h \nu_h, f_c R_{h'} \nu_{h'}\}, \\
A_h(\nu) &= \{s \in \tilde{S} | h R_s \nu_s\} \\
&\quad \cup \{m_c \in M | \exists h' \in \tilde{H}, (h, h') R_c \nu_c, f_c R_{h'} \nu_{h'}\} \\
&\quad \cup \{f_c \in F | \exists h' \in \tilde{H}, (h', h) R_c \nu_c, m_c R_{h'} \nu_{h'}\}.
\end{aligned}$$

We define a function $T : \Phi \to \Phi$ by $T_i(\nu) = \max_{R_i} A_i(\nu)$ for any $i \in N$.[7]

Functions similar to T have been used to study matching problems by Adachi (2000), Echenique and Oviedo (2004, 2006), Fleiner (2003), Hatfield and Milgrom (2005) and Ostrovsky (2005).

We present a part of our main result as the following Lemma.

Lemma 1. *ν is a matching if ν is a fixed point of T.*

Proof. Since ν is a prematching, we show that conditions (1), (2) and (3) of Definition 1 hold if $\nu = T\nu$.

(1) Suppose $\nu_s = h \in H$. Then $s \in A_h(\nu)$ by definition of $A_h(\nu)$. Therefore $\nu_h = T_h(\nu) R_h s$. On the other hand, $T_s(\nu) = \nu_s = h$ implies $h \in A_s(\nu)$ and hence $s R_h \nu_h$. Since R_h is a strict preference, we obtain $\nu_h = s$.

(2) Suppose $\nu_c = (h, h')$ and $h \in H$. Then

$$(h, h') = \nu_c = T_c(\nu) = \max_{R_c} A_c(\nu)$$

implies $m_c R_h \nu_h$ and $f_c R_{h'} \nu_{h'}$. The latter condition and $\nu_c = (h, h')$ imply $m_c \in A_h(\nu)$ by definition of $A_h(\nu)$. Therefore $\nu_h = T_h(\nu) R_h m_c$. Since R_h is a strict preference, we obtain $\nu_h = m_c$. If $\nu_c = (h, h')$ and $h' \in H$, we obtain $\nu_{h'} = f_c$ by a similar argument.

(3) Suppose $\nu_h = s \in S$. Then we obtain $\nu_s = h$ by an argument similar to (1). Suppose $\nu_h = m_c \in M$. Let $\nu_c = T_c(\nu) = (h'', h')$. Then $(h, h') \in A_c(\nu)$ since $T_c(\nu) \in A_c(\nu)$ by definition of $T_c(\cdot)$. Therefore the set $(\{h\} \times \tilde{H}) \cap A_c(\nu)$ is nonempty. Let $h''' \in \tilde{H}$ be the hospital (or $\emptyset$) such that $(h, h''') \in A_c(\nu)$ and $(h, h''') R_c (h, \tilde{h})$ for any $(h, \tilde{h}) \in A_c(\nu)$; such h''' exists and is unique since the set $(\{h\} \times \tilde{H}) \cap A_c(\nu)$ is nonempty

[7] For any finite set X and a strict preference relation R_i, $\max_{R_i} X$ denotes an element x of X such that $x R_i y$ for any $y \in X$.

and R_c is a strict preference. By definition of T, we have

$$\nu_c = T_c(\nu) R_c (h, h'''). \tag{1}$$

On the other hand, $T_h(\nu) = \nu_h = m_c$ implies that $m_c \in A_h(\nu)$. Therefore by definition of $A_h(\nu)$ there exists $\hat{h} \in \tilde{H}$ such that

$$(h, \hat{h}) R_c \nu_c \tag{2}$$

and $f_c R_{\hat{h}} \nu_{\hat{h}}$. Since $\nu_h = m_c$ by assumption and $f_c R_{\hat{h}} \nu_{\hat{h}}$ as noted above, definition of $A_c(\nu)$ implies

$$(h, \hat{h}) \in A_c(\nu). \tag{3}$$

Definition of h''' and relations (2) and (3) imply $(h, h''') R_c (h, \hat{h}) R_c \nu_c$. Since R_h is a strict preference, this relation and relation (1) imply $\nu_c = (h, h''')$, the desired relation. If $\nu_h = f_c \in F$, we obtain $\nu_c = (h', h)$ with $h' \in \tilde{H}$ by a similar argument. □

The following is the main result, corresponding to Theorem 3.1 of Echenique and Yenmez (2006).

Theorem 1. *ν is a fixed point of T if and only if it is a stable matching.*

Proof. **"Only if" part.** First, we show that any fixed point ν of T is a stable matching. Note first that ν is a matching by Lemma 1. Clearly ν is individually rational since it is a fixed point of T. Suppose that $m_c R_h \nu_h$ and $f_c R_{h'} \nu_{h'}$. This implies that $(h, h') \in A_c(\nu)$. Therefore $\nu_c = T_c(\nu) R_c (h, h')$, showing that (c, h, h') does not block ν. By a similar argument no $(s, h) \in S \times H$ blocks ν, showing that ν is stable.

"If" part. Next, we show that if ν is stable, then $T\nu = \nu$. First, consider $s \in S$. Since ν is a matching, $\nu_s = h \in H$ implies $\nu_h = s$, so $\nu_s \in A_s(\nu)$ (if $\nu_s = \emptyset$, then $\nu_s \in A_s(\nu)$ holds trivially). Since ν is individually rational, $\nu_s R_s \emptyset$. For any $h \in A_s(\nu) \cap H$, we have $s R_h \nu_h$ by definition of $A_s(\nu)$ and stability implies $\nu_s R_s h$. Therefore $T_s(\nu) = \nu_s$. Similar arguments establish that $T_c(\nu) = \nu_c$ and $T_h(\nu) = \nu_h$ for every $c \in C$ and $h \in H$, completing the proof. □

Let us introduce a partial order on Φ. We write $\nu \succeq \nu'$ if and only if $\nu_i R_i \nu'_i$ for every $i \in N$. We say that a function $F : \Phi \to \Phi$ is **monotone decreasing** with respect to $\succeq$ if $F(\nu) \preceq F(\nu')$ whenever $\nu \succeq \nu'$. The following Proposition is a counterpart of Lemma 4.3 of Echenique and Yenmez (2006).

Proposition 1. *T is monotone decreasing with respect to $\succeq$.*

Proof. Suppose $\nu \succeq \nu'$. Since $\nu_j R_j \nu'_j$ for every $j \in N$, we have $A_i(\nu) \subseteq A_i(\nu')$ for any $i \in N$. Therefore $T_i(\nu') = \max_{R_i} A_i(\nu') R_i \max_{R_i} A_i(\nu) = T_i(\nu)$ for every $i \in N$, completing the proof. □

Echenique and Yenmez (2006) develop an algorithm to find all the stable matchings in a given many-to-one matching market in which doctors have preferences over their colleagues as well as hospitals they attend. Their method, adapting the algorithm of Echenique (2003), relies on the fact that the set of stable matchings is equivalent to fixed points of a monotone decreasing function similar to T in this paper. Utilizing Theorem 1 and Proposition 1, one can construct an algorithm similar to theirs to find all the stable matchings in a given market with couples. The description of the algorithm and the proof that it finds all the stable matchings are omitted, as they are straightforward modifications of Echenique and Yenmez (2006).

References

Abdulkadiroğlu, A., P. A. Pathak, and A. E. Roth (2005): "The New York City High School Match," *American Economic Review Papers and Proceedings*, 95, 364–367.

Abdulkadiroğlu, A., P. A. Pathak, A. E. Roth, and T. Sönmez (2005): "The Boston Public School Match," *American Economic Review Papers and Proceedings*, 95, 368–372.

Abdulkadiroğlu, A., and T. Sönmez (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

Adachi, H. (2000): "On a characterization of stable matchings," *Economics Letters*, 68, 43–49.

Balinski, M., and T. Sönmez (1999): "A tale of two mechanisms: student placement," *Journal of Economic Theory*, 84, 73–94.

Dutta, B., and J. Masso (1997): "Stability of matchings when individuals have preferences over colleagues," *Journal of Economic Theory*, 75, 464–475.

Echenique, F. (2003): "Finding All Equilibria in Games with Strategic Complements," *Journal of Economic Theory*, forthcoming.

Echenique, F., and J. Oviedo (2004): "Core Many-to-One Matchings by Fixed Point Methods," *Journal of Economic Theory*, 115, 358–376.

——— (2006): "A theory of stability in many-to-many matching," *Theoretical Economics*, 1, 233–273.

Echenique, F., and M. B. Yenmez (2006): "A Solution to Matching with Preferences over Colleagues," *Games and Economic Behavior*, forthcoming.

Erdil, A., and H. Ergin (2006): "What's the Matter with Tie-Breaking? (Improving Efficiency in School Choice)," *mimeo.*

Fleiner, T. (2003): "A Fixed-Point Approach to Stable Matchings and Some Applications," *Mathematics of Operations Research*, 28, 103–126.

Gale, D., and L. S. Shapley (1962): "College admissions and the stability of marriage," *American Mathematical Monthly*, 69, 9–15.

Hatfield, J., and P. Milgrom (2005): "Matching with Contracts," *American Economic Review*, 95, 913–935.

Immorlica, N., and M. Mahdian (2005): "Marriage, Honesty, and Stability," *SODA 2005*, pp. 53–62.

Klaus, B., and F. Klijn (2005): "Stable matchings and preferences of couples," *Journal of Economic Theory*, 121, 75–106.

——— (2007): "Paths to Stability for Matching Markets with Couples," *Games and Economic Behavior*, 58, 154–171.

Klaus, B., F. Klijn, and J. Masso (2007): "Some Things Couples Always Wanted to Know about Stable Matchings (but were afraid to ask)," *Review of Economic Design*, forthcoming.

Klaus, B., F. Klijn, and T. Nakamura (2006): "Corrigendum: Stable Matchings and Preferences of Couples," mimeo.

Kojima, F., and P. A. Pathak (2006): "Incentives and stability in large two-sided matching markets," mimeo.

Ostrovsky, M. (2005): "Stability in Supply Chain Networks," .

Roth, A. E. (1984): "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory," *Journal of Political Economy*, 92, 991–1016.

Roth, A. E., and E. Peranson (1999): "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design," *American Economic Review*, 89, 748–780.

Roth, A. E., and M. A. O. Sotomayor (1990): *Two-sided matching: a study in game-theoretic modeling and analysis*. Econometric Society monographs, Cambridge.