

Anti-Coordination Games and Dynamic Stability*

Fuhito Kojima[†] Satoru Takahashi[‡]

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[†]Harvard University. kojima@fas.harvard.edu

[‡]Harvard University. stakahas@fas.harvard.edu

Abstract

We introduce the class of anti-coordination games. A symmetric two-player game is said to have the *anti-coordination property* if, for any mixed strategy, any worst response to the mixed strategy is in the support of the mixed strategy. Every anti-coordination game has a unique symmetric Nash equilibrium, which lies in the interior of the set of mixed strategies. We investigate the dynamic stability of the equilibrium in a one-population setting. Specifically we focus on the best response dynamic (BRD), where agents in a large population take myopic best responses, and the perfect foresight dynamic (PFD), where agents maximize total discounted payoffs from the present to the future. For any anti-coordination game we show (i) that, for any initial distribution, BRD has a unique solution, which reaches the equilibrium in a finite time, (ii) that the same path is one of the solutions to PFD, and (iii) that no path escapes from the equilibrium in PFD once the path reaches the equilibrium. Moreover we show (iv) that, in some subclasses of anti-coordination games, for any initial state, any solution to PFD converges to the equilibrium. All the results for PFD hold for any discount rate.

1 Introduction

What strategy is stable under what dynamic? It has been shown that an important role is played by the concept of evolutionarily stable strategy (ESS) and especially that of interior ESS.¹ An interior ESS is globally stable under various dynamics such as the replicator dynamic, the best response dynamic, and smoothed best response dynamics (Hofbauer [5, 6]; Hofbauer and Sandholm [7]).

One aspect of economic behavior, however, has been ignored in the study of ESS: forward-looking behavior under rational expectations. Economic agents take expectations into account in their decisions at least to some extent. Dynamics with expectations have been applied to economic analyses (Diamond and Fudenberg [1]; Kaneda [15]; Krugman [18]; Matsuyama [22]). Dynamic behavior may be qualitatively different between myopic and forward-looking dynamics. Given the importance of expectations in economics it is worth investigating the stability of action distributions in dynamics with expectations, but it is not known to date whether an ESS is dynamically stable in the presence of expectations.

Consider the following example. An economy consists of a continuum of producers. In each short interval of time, a number of firms enter the market and the same number of incumbents are randomly selected to exit from the market. Upon entrance, a firm chooses an industrial good $i = 1, \dots, n - 1$ or an agricultural good n , and produces one unit of the good in each unit of time. Switching costs are so high that a firm cannot change its decision until it is forced to exit.² Let x_i denote the fraction of firms producing good i and $x = (x_1, \dots, x_n)$. We normalize $\sum_i x_i = 1$. We assume that the per-period profit of producing good $i \neq n$ is given by $P_i(x) - c_i$, where $P_i(x) = b_i - d_i x_i - a_n \sum_{j \neq i, n} x_j$ with a_n, b_i, c_i , and d_i being constants. P_i is the inverse demand function for good i , which is subject to the substitution effect $-a_n \sum_{j \neq i, n} x_j$ from the other types of industrial goods. c_i is the unit cost of production. We assume $d_i > a_n > 0$. That is, industrial goods are imperfect substitutes to one another. The agricultural sector n is perfectly competitive, yielding zero profit. Define $a_i = d_i - a_n > 0$. Then the profit for $i \neq n$ is written as $-a_i x_i + a_n x_n + (b_i - c_i - a_n)$ since $\sum_{j \neq i, n} x_j = 1 - x_i - x_n$.

For simplicity, assume $b_i - c_i - a_n = 0$ for every $i \neq n$.³ Normalize the

¹See, for example, Fudenberg and Levine [2], Hofbauer and Sigmund [8, 9] and Weibull [29].

²An “entrant” can be the same entity as an “incumbent” which receives an opportunity to revise its action. Formal analysis is unchanged under this interpretation.

³Even without $b_i - c_i - a_n = 0$, some of our results (Propositions 3, 4 and 5) hold if the anti-coordination property introduced in this paper is satisfied. By Proposition 2, the

profit of producing $i = 1, \dots, n$ by subtracting $a_n x_n$. The normalized profit of $i = 1, \dots, n$ is given by $-a_i x_i$. This can be regarded as the average payoff when a firm is matched to another firm randomly drawn from the distribution x and these two firms play a symmetric two-player game with the following payoff matrix,

$$\begin{pmatrix} -a_1 & 0 & \cdots & 0 \\ 0 & -a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_n \end{pmatrix}. \quad (1)$$

The game expressed by (1) has a unique symmetric Nash equilibrium $x^* = (\lambda a_1^{-1}, \lambda a_2^{-1}, \dots, \lambda a_n^{-1})$, where $\lambda = (\sum_i a_i^{-1})^{-1}$. At state x^* , every good is produced by a positive fraction of the firms, and profits are equalized across goods.

Given the above environment, how does the production of each good change over time? Assume for now that every entrant chooses a good which maximizes the current profit. Such myopic behavior is modeled by the best response dynamic (BRD) introduced by Gilboa and Matsui [4] and Matsui [19]. BRD is a continuous-time dynamic in a large population of rational but myopic agents, each of whom takes one of the best responses to the current action distribution. Since x^* is not only a Nash equilibrium but also an interior ESS, the solution to BRD converges to x^* no matter how far the initial action distribution is from x^* (Hofbauer [5, 6]).⁴⁵ In our example, if firms maximize current profits upon entrance, then the distribution of goods produced in the market will approach x^* .

However, it may be unrealistic to assume that firms are completely myopic. Since each firm has to produce the same type of good during its lifetime, it would form an expectation about the future market environment and choose a good that maximizes the total discounted profit from the present to the future under this expectation. Such forward-looking behavior is modeled

current example satisfies the anti-coordination property if and only if

$$\sum_{i \in B} a_i^{-1} e_i < 1 + \left(\sum_{i \in B} a_i^{-1} \right) \min_{j \notin B} e_j$$

for any nonempty subset $B \subsetneq \{1, \dots, n\}$, where $e_i = b_i - c_i - a_n$ for $i \neq n$ and $e_n = 0$.

⁴Note that a Nash equilibrium may not be stable in evolutionary dynamics even if it is the unique equilibrium. See Gaunersdorfer and Hofbauer [3], Jordan [13] and Shapley [25].

⁵Since payoff matrix (1) is not only negative definite but also symmetric, we may apply results on potential games (or partnership games). See Hofbauer [6].

by the perfect foresight dynamic (PFD) introduced by Matsui and Matsuyama [20]. Agents in PFD maximize total discounted payoffs under their expectations, and their expectations are of perfect foresight.

Behavior of PFD in payoff matrix (1) has not been fully understood. Although x^* is an interior ESS, an interior ESS has not been proved to be dynamically stable.⁶ Will the distribution of goods produced in our example tend to x^* when firms have perfect foresight? Motivated by example (1), we also ask a more general question: Under what conditions can we make clear predictions in PFD?

To answer the above questions, we introduce the notion of anti-coordination. A symmetric game is said to have the *anti-coordination property* if, for any mixed strategy, any worst response to the mixed strategy lies in the support of the mixed strategy. In other words, a pure strategy is one of the worst responses only if the strategy is chosen by a positive fraction of the agents in the population. (1) is an example of an anti-coordination game. An anti-coordination game has a unique symmetric Nash equilibrium, which is in the interior of the set of mixed strategies. This symmetric Nash equilibrium may not be an ESS, and a game with an interior ESS may not have the anti-coordination property.

Although the anti-coordination property and ESS are not nested to each other, both concepts capture the same idea of “negative feedback.” Suppose that incumbents take a mixed strategy x in a society, and that a small number of entrants come to the society, taking a different mixed strategy y . In the society, both the incumbents and the entrants get payoffs from playing a game against a convex combination of x and y . If x is an ESS, by its definition, the average payoff of the entrants is strictly lower than that of the incumbents. By contrast, if the game has the anti-coordination property and x is its unique symmetric Nash equilibrium, the support of y includes all the pure-strategy worst responses to y . Those worst responses are also the worst responses to the convex combination of x and y because x is an interior equilibrium. Therefore, although the entrants may yield a higher payoff than the incumbents *on average* (hence x may not be an ESS), *some* of the entrants’ pure strategies yield the lowest payoff.

In an anti-coordination game, we investigate dynamic stability of its symmetric Nash equilibrium in a one-population setting. We obtain the following results for BRD and PFD. (i) For BRD there is a unique solution for each initial state, which reaches the equilibrium in a finite time. (ii) The unique path in BRD is also a solution to PFD. No matter how far from the equilibrium

⁶As in Footnote 5, we may apply results on potential games. See Hofbauer and Sorger [10] and Footnote 19.

the current action distribution in the society is, people outside the society cannot exclude the possibility that the action distribution will be close to the equilibrium in the future. (iii) No path escapes from the equilibrium in PFD once the path reaches the equilibrium. Therefore, if an outside observer sees that the action distribution in the society is in equilibrium, then she can predict that the action distribution will stay there forever without any reference to people's expectations in the society. (iv) In some subclasses of anti-coordination games including (1), for any initial state, any solution to PFD converges to the equilibrium. Thus one can predict that the limit of the action distribution in the future is the equilibrium, even if she knows neither the current action distribution nor people's expectations. One can therefore predict that in example (1) the economy will end up producing goods as prescribed by x^* . All the results for PFD hold for any discount rate.

Matsui and Oyama [21] show that the unique symmetric Nash equilibrium is stable under PFD in a 2×2 anti-coordination game. Our results extend their result to general $n \times n$ anti-coordination games.

PFD is formalized by Matsui and Matsuyama [20] to investigate equilibrium selection in generic 2×2 coordination games. They show (a) that there exists a solution to PFD from the risk-dominated equilibrium to the risk-dominant equilibrium if the rate of pure time discounting is below a sufficiently small positive number, and (b) if the rate of pure time discounting is positive, there exists no solution that escapes from the risk-dominant equilibrium. Based on (a) and (b), they argue that the risk-dominant equilibrium should be selected. Their argument about equilibrium selection is generalized in various directions (Hofbauer and Sorger [10, 11]; Kim [16]; Kojima [17]; Oyama [23]; Oyama et al. [24]; Takahashi [27]; Tercieux [28]). Our paper is different from these papers in two respects. First, all of our results on dynamic stability are independent of the discount rate while results in the previous literature are not.⁷ Second, one of the stability notions employed in this paper is much stronger than in the previous literature. Previous papers show that, for any initial state, *some* solution converges to a particular equilibrium (as in (a)), and a stationary path at the equilibrium is the only solution to PFD provided that the initial state is at the equilibrium (as in (b)). We show in result (iv) that, for several cases including (1), for *any* initial state, *any* solution to PFD converges to the Nash equilibrium.

⁷All the games analyzed in the previous literature on PFD have multiple equilibria while anti-coordination games have unique equilibria. Note that if there are more than one strict Nash equilibria, then there exists a solution to PFD escaping from any Nash equilibrium for a sufficiently small discount rate (here we allow the rate of pure time discounting to be negative). Therefore dynamic stability independent of the discount rate is obtainable in none of the games studied in the previous literature.

The rest of this paper is organized as follows. Section 2 introduces anti-coordination games. Section 3 shows that the symmetric Nash equilibrium in any anti-coordination game is globally stable under BRD. Section 4 introduces PFD, and investigates the stability of the equilibrium under the dynamic. Section 5 discusses the assumption of piecewise linearity in the two dynamics. Section 6 concludes.

2 Anti-Coordination Games

Consider a symmetric two-player game $G = (A, u)$, where A is the nonempty finite set of pure actions, and $u = (u_{ij})$ is the payoff matrix. u_{ij} is the payoff of an agent choosing action $i \in A$ against action $j \in A$. The set of mixed actions is denoted by $\Delta = \{x \in \mathbb{R}^A \mid x_i \geq 0 \text{ for all } i \in A, \sum_{i \in A} x_i = 1\}$. For each $x \in \Delta$, $\text{supp}(x) = \{i \in A \mid x_i > 0\}$ is the support of x , $\text{br}(x) = \arg \max_{i \in A} \sum_{j \in A} u_{ij}x_j$ is the set of best responses to x in pure actions, and $\text{wr}(x) = \arg \min_{i \in A} \sum_{j \in A} u_{ij}x_j$ is the set of worst responses to x in pure actions. $x \in \Delta$ is a *symmetric Nash equilibrium* if $\text{supp}(x) \subseteq \text{br}(x)$.

Definition 1. G has the *anti-coordination property* if $\text{wr}(x) \subseteq \text{supp}(x)$ for any $x \in \Delta$.

$G = (A, u)$ has the anti-coordination property if and only if $(A, -u)$ has the total bandwagon property (Kandori and Rob [14]). Payoff matrix (1) in the introduction (see also Example 6) and the hawk-dove game (Example 4) are examples of anti-coordination games.

Proposition 1. *Every anti-coordination game has a unique symmetric Nash equilibrium. The equilibrium is in the interior of Δ .*

Proof. The existence of a symmetric Nash equilibrium is clear. For any symmetric Nash equilibrium x , the anti-coordination property implies

$$\text{wr}(x) \subseteq \text{supp}(x) \subseteq \text{br}(x).$$

Since $\text{wr}(x) \subseteq \text{br}(x)$ only if the two sets are equal to A , we have $\text{supp}(x) = A$.

Suppose that there are two different interior symmetric Nash equilibria x and y . Since the game is a two-player game, the payoff function is linear in the opponent's mixed strategy. Therefore, any affine combination of x and y , i.e., any point on the line connecting x and y , is also a symmetric Nash equilibrium. There are two intersections of the line and the boundary of Δ , each of which is a boundary symmetric Nash equilibrium. This contradicts the fact that every symmetric Nash equilibrium is in the interior of Δ . \square

Let x^* denote the unique symmetric Nash equilibrium of G . For any nonempty subset B of A , let $G(B)$ be the *restricted game* of G in which the players choose actions only from B . If G has the anti-coordination property, then any restricted game of G also has the same property, and hence has a unique symmetric Nash equilibrium, whose support is B . The symmetric Nash equilibrium of $G(B)$ is denoted by $x^*(B)$.

Proposition 2. *G has the anti-coordination property if and only if $G(B)$ has the anti-coordination property and $\text{wr}(x^*(B)) = B$ for any $B \subsetneq A$.⁸*

Proof. See Appendix. □

We can use Proposition 2 inductively on the size of restricted games to characterize the anti-coordination property. See the next example.

Example 1. Consider an arbitrary 3×3 payoff matrix on $A = \{1, 2, 3\}$,

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}.$$

We will give a necessary and sufficient condition for this payoff matrix to have the anti-coordination property.

First, we consider a restricted game $G(\{i\})$ for each $i \in A$. $G(\{i\})$ is obviously an anti-coordination game, and pure strategy i is the unique symmetric Nash equilibrium $x^*(\{i\})$. Then the condition that $\text{wr}(x^*(\{i\})) = \{i\}$ for each $i \in A$ is written as

$$u_{ii} < u_{ji} \quad \text{for any } i \neq j, \tag{2}$$

i.e., each diagonal component is smaller than any other components in the same column. By Proposition 2, we know that each 2×2 restricted game $G(\{i, j\})$ is an anti-coordination game if (2) is satisfied. Under this condition, by Proposition 1, $G(\{i, j\})$ has a unique symmetric Nash equilibrium $x^*(\{i, j\})$, which is given by

$$x_i^*(\{i, j\}) = \frac{u_{ij} - u_{jj}}{u_{ij} + u_{ji} - u_{ii} - u_{jj}}, \quad x_j^*(\{i, j\}) = \frac{u_{ji} - u_{ii}}{u_{ij} + u_{ji} - u_{ii} - u_{jj}}.$$

Then the condition that $\text{wr}(x^*(\{i, j\})) = \{i, j\}$ for each $i \neq j$ is equivalent to

$$u_{ij}u_{ji} - u_{ii}u_{jj} < u_{ki}(u_{ij} - u_{jj}) + u_{kj}(u_{ji} - u_{ii}) \quad \text{for any distinct } i, j, k. \tag{3}$$

Thus the 3×3 game G has the anti-coordination property if and only if (2) and (3) hold.

⁸Here $x^*(B)$ is regarded as an element of Δ , i.e., a mixed strategy in the unrestricted game; $\text{wr}(\cdot)$ is the worst response correspondence in the unrestricted game.

The unique symmetric Nash equilibrium of an anti-coordination game may not be evolutionarily stable, and a game with an interior ESS may not have the anti-coordination property. For example, the following payoff matrix

$$\begin{pmatrix} 0 & a & b \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

has the anti-coordination property if and only if $a > 0$, $b > 0$, and $a + b > 1$, whereas it has an interior ESS if and only if $a + b > 1$ and $4(a + b + 1) > (a - b)^2$. $(a, b) = (8, 1)$ satisfies the former condition only; $(a, b) = (2, -0.5)$ satisfies the latter condition only.

Similarly, an anti-coordination game may not be a potential game and vice versa.

3 The Best Response Dynamic

Consider the best response dynamic (BRD) in game G (Gilboa and Matsui [4], Matsui [19]):

$$\phi: [0, \infty) \rightarrow \Delta, \tag{BRD-0}$$

$$\phi(0) = x, \tag{BRD-1}$$

$$\frac{d^+ \phi}{dt}(t) = \alpha(t) - \phi(t), \tag{BRD-2}$$

$$\text{supp}(\alpha(t)) \subseteq \text{br}(\phi(t)). \tag{BRD-3}$$

A microfoundation of BRD is as follows. There is one large population of agents. The action distribution at time t is represented by a mixed action $\phi(t) \in \Delta$ (BRD-0). x is the initial action distribution (BRD-1). At each moment in time, an agent is matched randomly with another in the same population and these two agents play G . An infinitesimal fraction dt of randomly chosen agents change their actions between periods t and $t + dt$. The distribution of actions chosen in time interval $[t, t + dt)$ is proportional to $\alpha(t)$ (BRD-2), and every pure action chosen by a positive fraction of the agents is one of the best responses to the current action distribution (BRD-3).

There exists at least one solution to (BRD-0)–(BRD-3) for any stage game G and any initial state $x \in \Delta$ (Hofbauer [5]).⁹ However, there may be multiple solutions.

⁹There is a technical issue concerning the relation between the existence of a solution and the class of paths considered. See Section 5 for details.

Example 2. Consider the following payoff matrix on $A = \{1, 2\}$,

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

which does not have the anti-coordination property. There are three symmetric Nash equilibria: $x^1 = (x_1^1, x_2^1) = (1, 0)$ (pure strategy 1), $x^2 = (1/3, 2/3)$, and $x^3 = (0, 1)$ (pure strategy 2). Suppose that $\phi(0) = x^2$. Since x^2 is a symmetric Nash equilibrium, the constant path at x^2 is a solution to BRD. There are, however, infinitely many other solutions as well. For any $T \geq 0$, let ϕ^T be the path given by

$$\phi^T(t) = \begin{cases} x^2 & (0 \leq t < T), \\ (1 - e^{T-t})x^1 + e^{T-t}x^2 & (t \geq T), \end{cases} \quad \alpha^T(t) = \begin{cases} x^2 & (0 \leq t < T), \\ x^1 & (t \geq T). \end{cases}$$

ϕ^T stays at the initial state x^2 for T periods, and then moves toward x^1 . It is easy to see that ϕ^T is a solution to BRD. Similarly, any path which stays at x^2 for a while and then moves toward x^3 is also a solution.

More generally, if $G(\text{br}(x))$ has multiple symmetric Nash equilibria, then there are multiple BRD solutions starting at x .¹⁰

Example 3. Consider the following payoff matrix on $A = \{1, 2, 3\}$,

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

This game has the anti-coordination property. The unique symmetric Nash equilibrium is $x^* = (x_1^*, x_2^*, x_3^*) = (1/3, 1/3, 1/3)$. We will see that path ϕ from initial state x depicted in Figure 1 satisfies (BRD-0)–(BRD-3). The initial state x lies in the region where strategy 3 is the unique best response. Hence the path heads toward strategy 3 until it reaches point P , where strategies 2 and 3 become indifferent. At this moment half the population begins to take strategy 2 and the rest begins to take strategy 3. At the aggregate level, the path kinks at P and moves toward Q . After a finite time the path reaches x^* , where all three strategies are indifferent. Each strategy is chosen by one third of the population and the path stays at rest afterwards.

Four features deserve comment here. First, x^* is globally stable. That is, from any initial state x , there exists a unique path ϕ satisfying (BRD-0)–(BRD-3), which converges to x^* . Second, ϕ is not differentiable. It has a

¹⁰Conversely, the uniqueness of a symmetric Nash equilibrium in $G(\text{br}(x))$ is sufficient for BRD to have a locally unique linear solution from x . This condition is no longer sufficient, however, once we drop the assumption of linearity (Hofbauer [5, Footnote 10]).

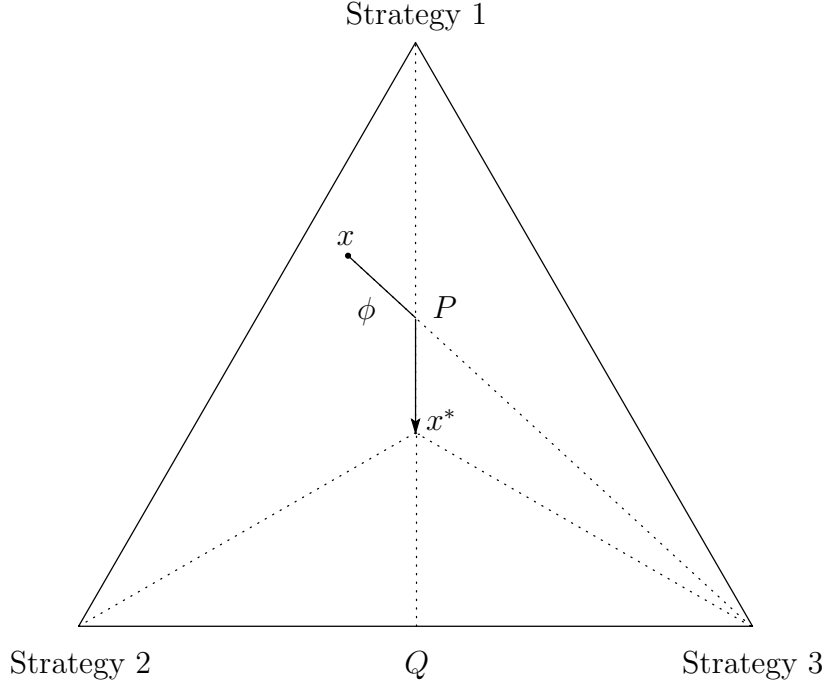


Figure 1: Example 3

kink (turning point) at P , where the fraction of agents choosing each action changes suddenly. Third, nevertheless, ϕ is right differentiable and piecewise linear. Fourth, $\alpha(t)$ in (BRD-2) may not be a pure strategy. That is, different agents may choose different actions at a point in time in general. After the path reaches P , strategies 2 and 3 are chosen simultaneously, and every strategy is taken by a positive fraction of the agents at x^* .

Henceforth, we restrict our analysis to piecewise linear solutions for mathematical convenience. In this paper, a path is said to be *piecewise linear* if it has a finite number of kinks in any bounded interval of time.¹¹

The next proposition generalizes the first feature in Example 3, showing that the symmetric Nash equilibrium of any anti-coordination game is globally stable under BRD. A kind of local stability is introduced by Matsui [19]. A symmetric Nash equilibrium x is said to be *socially stable with respect to BRD* if the constant path at x is the unique piecewise linear path from x under BRD. He shows that this concept is equivalent to *social stability against equilibrium entrants*, which is also called *robustness against symmetric equilibrium entrants* in Swinkels [26]. Global stability in the next proposition is stronger than this local stability.

¹¹We discuss the assumption of piecewise linearity in Section 5.

Proposition 3. *If G has the anti-coordination property, then, for any initial state $x \in \Delta$, there exists a unique piecewise linear path satisfying (BRD-0)–(BRD-3). The path arrives at x^* in a finite time and stays there afterwards.*

Proof. Let ϕ be a piecewise linear solution to (BRD-0)–(BRD-3). For every $t > 0$, there exist $t' > t$ and $\alpha \in \Delta$ such that $\alpha(s) = \alpha$ for all $s \in [t, t']$. Then we have

$$\text{supp}(\alpha) \subseteq \text{br}(\phi(s)) = \text{br}(c_\phi(s)\phi(t) + c_\alpha(s)\alpha)$$

for every $s \in [t, t']$, where $c_\phi(s) = e^{t-s}$ and $c_\alpha(s) = 1 - e^{t-s}$. When $s - t$ is positive but sufficiently small, we have

$$\text{br}(c_\phi(s)\phi(t) + c_\alpha(s)\alpha) = \text{br}(\alpha \mid \text{br}(\phi(t))),$$

where $\text{br}(y \mid B) = \arg \max_{i \in B} \sum_j u_{ij}y_j$ is the set of best responses to y within B . Since $\text{supp}(\alpha) \subseteq \text{br}(\alpha \mid \text{br}(\phi(t)))$, α is a symmetric Nash equilibrium of the restricted game $G(\text{br}(\phi(t)))$. By the anti-coordination property, we have $\alpha(t) = \alpha = x^*(\text{br}(\phi(t)))$. Thus $\alpha(\cdot)$ is uniquely determined by this construction.

Suppose that $\phi(t) \neq x^*$. Since $\alpha = x^*(\text{br}(\phi(t))) \neq x^*$, we have $\alpha(s) \neq \alpha$ for some $s > t$. (If $\alpha(s) = \alpha$ for any $s > t$, then $\phi(t) \rightarrow \alpha$ as $t \rightarrow \infty$, and $\text{supp}(\alpha) \subseteq \text{br}(\alpha)$ by the upper hemicontinuity of $\text{br}(\cdot)$. This contradicts $\alpha \neq x^*$.) Let t'' be the nearest kink after time t . Note that $\alpha(s) = \alpha$ for any $s \in [t, t'')$ and $\alpha(t'') \neq \alpha$. Since $\text{br}(\phi(s)) \supseteq \text{supp}(\alpha) = \text{br}(\phi(t))$ for any $s \in [t, t'')$, we have $\text{br}(\phi(t'')) \supseteq \text{br}(\phi(t))$ by the upper hemicontinuity of $\text{br}(\cdot)$. Since $x^*(\text{br}(\phi(t''))) = \alpha(t'') \neq \alpha = x^*(\text{br}(\phi(t)))$, we have $\text{br}(\phi(t'')) \neq \text{br}(\phi(t))$. Therefore $\text{br}(\phi(t))$ weakly increases in t in the set inclusion order, and strictly increases in a finite time until $\phi(t) = x^*$ is established. Therefore, ϕ arrives at x^* in a finite time and stays at x^* afterwards. \square

Global stability also follows from Hofbauer [5, Theorem 5.1.1]. He defines

$$V(x) = \max_i \sum_j u_{ij}x_j - w_{\mathbf{B}}(x),$$

where \mathbf{B} is the set of mixed strategies b such that $\text{supp}(b) \subsetneq A$ and every pure strategy in $\text{supp}(b)$ is indifferent against b , and

$$w_{\mathbf{B}}(x) = \max \left\{ \sum_{b \in \mathbf{B}} \sum_{i,j} u_{ij}b_i b_j \lambda^b \mid \lambda^b \geq 0, \sum_{b \in \mathbf{B}} \lambda^b = 1, \sum_{b \in \mathbf{B}} b \lambda^b = x \right\}.$$

He shows that if there exists $p \in \Delta$ with $\sum_{i,j} u_{ij}p_i p_j > \sum_{i,j} u_{ij}b_i b_j$ for all $b \in \mathbf{B}$, then V is a global Lyapunov function for BRD, which decreases

except at x^* . This implies the global stability of x^* . It is easy to see that any anti-coordination game satisfies the above condition.¹² Note that Hofbauer's result applies to a broader class of games including games with interior ESS.

For anti-coordination games, however, Proposition 3 gives a sharper prediction than Hofbauer's theorem in two respects. First, the piecewise linear path satisfying (BRD-0)–(BRD-3) is shown to be unique. Second, the path is constructed explicitly. This construction turns out to be useful when we show in Proposition 4 that the same path is a solution to PFD as well.

4 The Perfect Foresight Dynamic

Consider the perfect foresight dynamic (PFD) in game G (Matsui and Matsuyama [20]):

$$\phi: [0, \infty) \rightarrow \Delta, \tag{PFD-0}$$

$$\phi(0) = x, \tag{PFD-1}$$

$$\frac{d^+\phi}{dt}(t) = \alpha(t) - \phi(t), \tag{PFD-2}$$

$$\pi(t) = r \int_t^\infty e^{r(t-s)} \phi(s) ds, \tag{PFD-3}$$

$$\text{supp}(\alpha(t)) \subseteq \text{br}(\pi(t)). \tag{PFD-4}$$

PFD is different from BRD only in one respect. Unlike in BRD, agents in PFD maximize not current payoffs but total discounted payoffs from the present to the future under their expectations, which are assumed to be of perfect foresight. If an agent takes action i at period t , her total payoff is given by

$$r \int_t^\infty e^{r(t-s)} \left(\sum_j u_{ij} \phi_j(s) \right) ds = \sum_j u_{ij} \left(r \int_t^\infty e^{r(t-s)} \phi_j(s) ds \right) = \sum_j u_{ij} \pi_j(t).$$

Therefore, to maximize the total payoff, each agent chooses one of the best responses to $\pi(t)$, the discounted time average of the action distributions

¹²In an anti-coordination game, we have $\mathbf{B} = \{x^*(B) \mid B \subsetneq A\}$, that is, the set of symmetric Nash equilibria of strictly restricted games. Taking any totally mixed strategy as p , for instance, we can show $\sum_{i,j} u_{ij} p_i b_j > \sum_{i,j} u_{ij} b_i b_j$ for any $b = x^*(B)$ because any pure strategy outside B gives a higher payoff against $x^*(B)$ than any pure strategy inside B does.

from the present to the future (PFD-3, PFD-4). $r > 0$ is called the effective discount rate.¹³

Similarly to BRD, PFD has at least one solution for any stage game G and any initial state $x \in \Delta$, but may have multiple ones in general.¹⁴

Although PFD is close to BRD for sufficiently large r , a solution to BRD is not necessarily a solution to PFD. In Example 2, for instance, ϕ^T is a solution to PFD if and only if $T = 0$ or ∞ (the constant path at x^2). If $0 < T < \infty$, then ϕ^T is not a solution to PFD for any $r > 0$ because taking x^2 in period $[0, T)$ is not a best response given the expectation that the society will move toward x^1 after time T . In contrast, the next proposition shows that, in an anti-coordination game, the unique solution to BRD constructed in Proposition 3 is also a solution to PFD for any discount rate.

Proposition 4. *If G has the anti-coordination property, then, for any initial state $x \in \Delta$, there exists a piecewise linear path satisfying (PFD-0)–(PFD-4) which converges to x^* .*

Proof. Let ϕ be the path constructed in Proposition 3. We will show that ϕ also satisfies (PFD-4). Since $\text{br}(\phi(s)) \supseteq \text{br}(\phi(t))$ for any $s \geq t$, we have $\text{br}(\pi(t)) \supseteq \text{br}(\phi(t))$ by (PFD-3). Therefore (BRD-3) implies (PFD-4). \square

Proposition 4 shows that PFD has a solution from any state to x^* , i.e., x^* is *globally accessible* (Matsui and Matsuyama [20]). No matter how far from x^* the initial state is, it is possible for the action distribution in the society to arrive at x^* .

The existence of such a solution, however, does not imply that the society *always* reaches x^* . For PFD typically entails serious multiplicity of solutions, the dynamic may have a solution which does not converge to a Nash equilibrium. Moreover, there may be a path which escapes even from a strict Nash equilibrium. For example, consider Example 2 again. Note that $x^1 = (1, 0)$ and $x^3 = (0, 1)$ are strict Nash equilibria. Let $\phi(0) = x^1$. Then the constant path at x^1 is the unique solution to BRD, which is also a solution to PFD for any $r > 0$. For $0 < r \leq 1/2$, however, the following path

$$\phi(t) = e^{-t}x^1 + (1 - e^{-t})x^3, \quad \alpha(t) = x^3$$

¹³In most of the literature on PFD, r is assumed to be larger than one, where r is composed of the arrival rate of revision opportunities and the rate of pure time discounting θ . Since we have normalized the former to one, we have $r = 1 + \theta$. $r > 1$ is equivalent to $\theta > 0$ in this setup. In the presence of population growth or technological progress, however, r may be smaller than one even if θ is positive. Our analysis applies as long as $r > 0$.

¹⁴As in Footnote 9, see Section 5 for a class of paths that is large enough to guarantee the existence of solutions.

is also a solution to PFD. Similarly, for $\phi(0) = x^3$, the constant path at x^3 is a solution to PFD for any $r > 0$, and $\phi(t) = (1 - e^{-t})x^1 + e^{-t}x^3$ is another solution to PFD for $0 < r \leq 2$ (Matsui and Matsuyama [20]). In general, PFD has a solution escaping from x for sufficiently small $r > 0$ if there is a strict symmetric Nash equilibrium other than x . Therefore, for the constant path at some state to be the unique solution starting at the state for any $r > 0$, it is necessary for the stage game to have at most one strict symmetric Nash equilibrium.

Then what condition is sufficient? We will show that the anti-coordination property is sufficient for the constant path at x^* to be the unique solution originating at x^* for any $r > 0$ (within the class of piecewise linear paths). We first show the following lemma, which claims that $\alpha(t)$ does not include any myopic worst response to the current action distribution $\phi(t)$ unless $\phi(t)$ is equal to x^* . ($\alpha(t)$ may not be a myopic best response to $\phi(t)$.) This lemma is powerful, for we obtain a restriction on $\alpha(t)$ without any reference to the future behavior $\phi(s)$ for $s > t$.

Lemma 1. *If G has the anti-coordination property and ϕ is a piecewise linear path satisfying (PFD-0)–(PFD-4), then, for any $t \geq 0$, either $\phi(t) = x^*$ or $\text{supp}(\alpha(t)) \cap \text{wr}(\phi(t)) = \emptyset$ holds.*

Proof. Suppose $\phi(t) \neq x^*$ and $\text{supp}(\alpha(t)) \cap \text{wr}(\phi(t)) \neq \emptyset$ for some t . Let $\alpha = \alpha(t)$. Then there exists $t' > t$ such that

$$\text{supp}(\alpha) \subseteq \text{br}(\pi(s)) = \text{br}(c_\pi(s)\pi(t) - c_\phi(s)\phi(t) - c_\alpha(s)\alpha) \quad (4)$$

for every $s \in [t, t')$, where

$$c_\pi(s) = e^{r(s-t)}, \quad c_\phi(s) = \frac{r(e^{r(s-t)} - e^{t-s})}{1+r}, \quad c_\alpha(s) = \frac{e^{r(s-t)} + re^{t-s}}{1+r} - 1.$$

Since $c_\pi(\cdot)$, $c_\phi(\cdot)$, and $c_\alpha(\cdot)$ are linearly independent in the space of real-valued functions on $[t, t')$, (4) implies that every action in $\text{supp}(\alpha)$ is indifferent against α . By the anti-coordination property, we have $\text{supp}(\alpha) = \text{wr}(\alpha)$.

Let t'' be the nearest kink after time t . (If there is no kink after time t , skip to the last paragraph of this proof.) Because of the upper hemicontinuity of $\text{br}(\cdot)$, (4) holds also for $s = t''$. Since $c_\pi(t'')$, $c_\phi(t'')$, and $c_\alpha(t'')$ are all positive, the set of best responses to $\pi(t'')$ is given by the intersection of $\text{br}(\pi(t))$, $\text{wr}(\phi(t))$, and $\text{wr}(\alpha)$ if the intersection is nonempty. Since we have $\text{wr}(\alpha) = \text{supp}(\alpha) \subseteq \text{br}(\pi(t))$ and $\text{supp}(\alpha) \cap \text{wr}(\phi(t)) \neq \emptyset$, the intersection is actually nonempty and equal to $\text{supp}(\alpha) \cap \text{wr}(\phi(t))$. Therefore,

$$\text{supp}(\alpha(t'')) \subseteq \text{br}(\pi(t'')) = \text{supp}(\alpha) \cap \text{wr}(\phi(t)) = \text{wr}(\phi(s))$$

for any $t < s \leq t''$. Since $\phi(t) \neq x^*$, we have $\text{wr}(\phi(t'')) \subseteq \text{wr}(\phi(t)) \subsetneq A$, which implies $\phi(t'') \neq x^*$ as well. Also we have $\text{supp}(\alpha(t'')) \cap \text{wr}(\phi(t'')) = \text{supp}(\alpha(t'')) \neq \emptyset$. Hence we can continue the same argument for the next kink. Since we assume that there are only finitely many kinks in any bounded interval, we can inductively show that $\text{wr}(\phi(s))$ is a subset of $\text{supp}(\alpha)$ for any $s > t$ and decreasing in the set inclusion order in $s \geq t$.

Finally, by (PFD-3), $\pi(t)$ is a convex combination of $\phi(s)$ for $s \geq t$. Then we have $\text{wr}(\pi(t)) = \bigcap_{s \geq t} \text{wr}(\phi(s))$, which is a nonempty subset of $\text{supp}(\alpha)$. By (PFD-4), we have $\text{wr}(\pi(t)) \subseteq \text{supp}(\alpha) \subseteq \text{br}(\pi(t))$, hence $\text{wr}(\pi(t)) = \text{br}(\pi(t)) = A$. However, because of $\phi(t) \neq x^*$, we have $\text{wr}(\pi(t)) \subseteq \text{wr}(\phi(t)) \subsetneq A$, which is a contradiction. \square

As an immediate implication of Lemma 1, we obtain a kind of local stability under PFD.

Proposition 5. *If G has the anti-coordination property, then the constant path at x^* is the unique piecewise linear path satisfying (PFD-0)–(PFD-4) for $x = x^*$.*

Proof. Suppose that ϕ is a nonconstant piecewise linear solution from x^* . Then there exist t and t' with $0 \leq t < t'$ such that $\phi(t) = x^*$ and $\alpha(s) = \alpha \neq x^*$ for any $s \in [t, t']$. At time t' , we have $\phi(t') = e^{t-t'}x^* + (1 - e^{t-t'})\alpha \neq x^*$ and $\text{wr}(\phi(t')) = \text{wr}(\alpha) \subseteq \text{supp}(\alpha)$ by the anti-coordination property. This contradicts Lemma 1. \square

By Proposition 5, once the action distribution reaches x^* , it stays at rest forever. Therefore, an outside observer who currently sees the society at the equilibrium can predict that the society will stay there forever even if she is not informed of people's expectations in the society. Takahashi [27] calls this property *absorption in the discrete topology (d-absorption)*.

Proposition 5 and its proof are an extension of Matsui and Oyama [21, Lemma A.2]. They show the d-absorption of the unique symmetric Nash equilibrium in the hawk-dove game.¹⁵

Next, we discuss the global stability under PFD. Although we cannot obtain a general result in the class of anti-coordination games, we can show the global stability in several “simple” games. In these games, one can predict that the society will reach the equilibrium in the future. She needs to know

¹⁵Actually their Lemma A.2 is not for the hawk-dove game, but for a 3×3 game in which the hawk-dove game is “embedded” as a restricted game. However, as they remark below Proposition 7.2, we can use the same technique as in Lemma A.2 to show the d-absorption in the hawk-dove game.

neither the current state nor people’s expectations in order to make this prediction.¹⁶

Example 4. Consider the hawk-dove game on $A = \{1, 2\}$,

$$\begin{pmatrix} 0 & a \\ 1 - a & 0 \end{pmatrix}, \quad 0 < a < 1.$$

This game has the anti-coordination property. The unique symmetric Nash equilibrium is $x^* = (x_1^*, x_2^*) = (a, 1 - a)$. For any given initial state $x = (x_1, x_2)$, the path constructed in Proposition 4 is a solution to PFD. Here we will show that there is no other solution. Suppose that $\phi_1(t) > x_1^*$. Then $\text{wr}(\phi(t)) = \{1\}$, which implies $\alpha(t) = (0, 1)$ by Lemma 1. That is, ϕ moves toward pure strategy 2. Similarly, if $\phi_1(t) < x_1^*$, then ϕ moves toward pure strategy 1. If $\phi_1(t) = x_1^*$, then, by Proposition 5, ϕ stays at x^* forever. In summary, the hawk-dove game has a unique solution to PFD from any initial state, which arrives at the symmetric Nash equilibrium in a finite time.

Example 5. Consider the following payoff matrix on $A = \{1, 2, 3\}$,

$$\begin{pmatrix} 0 & a & 1 - a \\ 1 - a & 0 & a \\ a & 1 - a & 0 \end{pmatrix}, \quad \frac{1}{3} \leq a \leq \frac{2}{3}.$$

Since $0 < a < 1$, this is an anti-coordination game with the unique symmetric Nash equilibrium $x^* = (1/3, 1/3, 1/3)$.

We will show that any solution to PFD converges to x^* , and that the solution reaches x^* in a finite time if $1/3 < a < 2/3$. We divide the state space Δ into three regions $\Delta_i = \{x \in \Delta \mid i \in \text{wr}(x)\}$ for $i \in A$. Without loss of generality, we assume that $1/3 \leq a \leq 1/2$ and that the initial state x is in Δ_1 . See Figure 2.

First, notice that no solution ϕ crosses the border from Δ_i to $\Delta_{i-1} \setminus \Delta_i$.¹⁷ Otherwise, on the border $\Delta_i \cap \Delta_{i-1}$, the solution moves toward pure strategy $i + 1$ by Lemma 1. An increase in the proportion of $i + 1$ makes the other strategies better off. However, since $a \leq 1/2$, the payoff of $i - 1$ increases at least as much as that of i , which contradicts the direction in which the solution crosses the border.

¹⁶Krugman [18] and Matsuyama [22] investigate the issue of the so-called “history versus expectations” in dynamic economies with positive externalities. According to their analyses, history (the current state) is decisive for the ultimate economic outcome in some cases, and expectations matter in other cases. In Examples 4–6, in contrast, neither history nor expectations matter: Any solution originating at any initial state reaches the equilibrium.

¹⁷We take an element of A modulo 3. For example, $1 - 1 = 3$ and $3 + 1 = 1$.

Second, ϕ cannot stay forever in one region except at x^* . (If ϕ stays in $\Delta_i \setminus \{x^*\}$ forever, then $\phi_i(t) \rightarrow 0$ as $t \rightarrow \infty$ by Lemma 1, and finally ϕ gets out of Δ_i . This is a contradiction.) Therefore, by the first observation, ϕ goes from Δ_1 to $\Delta_2, \Delta_3, \Delta_1, \Delta_2, \dots$, that is, ϕ moves counterclockwise around x^* in Figure 2.

Third, define P_1, P_2, \dots as follows. Let P_1 be the intersection of the border $\Delta_1 \cap \Delta_2$ and the segment connecting pure strategies 1 and 2, and let P_k be the intersection of the border $\Delta_k \cap \Delta_{k+1}$ and the segment connecting P_{k-1} and pure strategy $k+1$ for $k \geq 2$. See Figure 2 for P_1, P_2 , and P_3 . By Lemma 1, ϕ moves toward a point between pure strategies $i \pm 1$ when ϕ stays in Δ_i . Therefore, when the solution crosses the k -th border, the pass point is between x^* and P_k .

Fourth, by a tedious computation, we have

$$d(P_k, x^*) = \begin{cases} \sqrt{2}/(3k) & \text{if } a = 1/3, \\ \frac{(a - 1/3)\sqrt{2(a^2 - a + 1/3)}}{a^{k+1}(1 - 2a)^{-k+1} - (a^2 - a + 1/3)} & \text{if } 1/3 < a < 1/2, \\ 1/\sqrt{6} & \text{if } a = 1/2 \text{ and } k = 1, \\ 0 & \text{if } a = 1/2 \text{ and } k \geq 2. \end{cases}$$

and hence $P_k \rightarrow x^*$ as $k \rightarrow \infty$.¹⁸ This fact, combined with the third observation, implies that any solution converges to x^* .

Fifth, note that, in Δ_k , the fraction of strategy k decreases at a speed bounded away from zero. When a solution moves from a boundary $\Delta_{k-1} \cap \Delta_k$ to the next boundary $\Delta_k \cap \Delta_{k+1}$, the fraction of strategy k can change by at most $d(P_k, x^*) + d(P_{k+1}, x^*)$. Therefore, there exists a constant $C > 0$ such that it takes time at most $C(d(P_k, x^*) + d(P_{k+1}, x^*))$ for any solution to move from $\Delta_{k-1} \cap \Delta_k$ to $\Delta_k \cap \Delta_{k+1}$. Therefore, if $1/3 < a \leq 1/2$, then any solution reaches x^* in a finite time since $\sum_{k=1}^{\infty} d(P_k, x^*) < \infty$.

Example 6. In the introduction, we investigated the following payoff matrix (1) on $A = \{1, \dots, n\}$,

$$\begin{pmatrix} -a_1 & 0 & \cdots & 0 \\ 0 & -a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_n \end{pmatrix}, \quad a_1 > 0, \dots, a_n > 0.$$

This game has the anti-coordination property. The unique symmetric Nash equilibrium is $x^* = (\lambda a_1^{-1}, \lambda a_2^{-1}, \dots, \lambda a_n^{-1})$, where $\lambda = (\sum_i a_i^{-1})^{-1}$.

¹⁸ $d(y, z) = \sqrt{\sum_i (y_i - z_i)^2}$ for $y, z \in \Delta$.

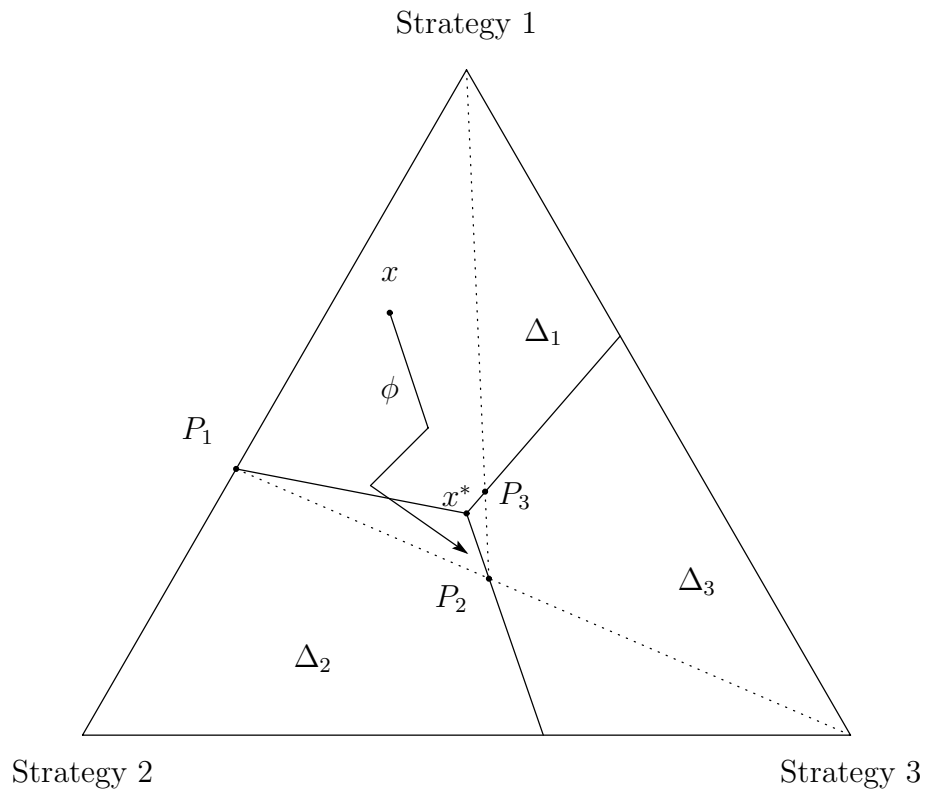


Figure 2: Example 5

We will show that any solution to PFD converges to x^* in a finite time. As in Example 5, we divide Δ into n regions $\Delta_i = \{x \in \Delta \mid i \in \text{wr}(x)\}$.

Similarly to Example 5, any solution ϕ crosses the border from Δ_i to $\Delta_j \setminus \Delta_i$ for $i \neq j$ unless $\phi(t) = x^*$ for some t . Let t_0 be the moment of crossing the border. Then the ratio $\phi_i(t)/\phi_j(t)$ is equal to a_j/a_i for $t = t_0$, and is below a_j/a_i for t slightly greater than t_0 . Therefore, $\phi_i(t)/\phi_j(t)$ is decreasing around $t = t_0$. This implies that $\alpha_j(t_0) > 0$, which contradicts Lemma 1.¹⁹

5 How Restrictive is Piecewise Linearity?

Our analysis has assumed that solutions to each dynamic are piecewise linear. This assumption simplifies our arguments in Proposition 3 for BRD; Lemma 1, Proposition 5, and Examples 4–6 for PFD. This section discusses the validity of this assumption.

A piecewise linear path is defined as a function which has a finite number of kinks in any bounded interval of time. Equivalently, $\alpha(\cdot)$ is a step function (with a finite number of values) on any bounded interval of time. The class of piecewise linear paths includes linear paths, paths that are kinked finitely often, and paths that are kinked infinitely often at $t = 1, 2, 3, \dots$, but excludes smoothly curved paths and paths that are kinked infinitely often at $t = 1, 1/2, 1/3, 1/4, \dots$.

The class of piecewise linear paths is also used in Matsui's [19] analysis of BRD. In this class, he gives a simple necessary and sufficient condition for a symmetric Nash equilibrium to be socially stable with respect to BRD.²⁰

¹⁹ The payoff matrix of Example 6 is symmetric, i.e., two players always get identical payoffs. For a symmetric payoff matrix, global accessibility in PFD (Proposition 4) is already obtained by Hofbauer and Sorger [10, Theorem 3] if the effective discount rate r is greater than but sufficiently close to 1. They also show in [10, Lemma 4] that any element of the ω -limit of each solution to PFD is a critical point of the potential function if $r > 1$. Since the ω -limit is connected and any connected component of critical points in Example 6 is a singleton, the ω -limit is a singleton, i.e., the solution is a convergent path. Because the limit is a Nash equilibrium under PFD, the solution converges to x^* . Hence global stability is also derived from Hofbauer and Sorger's analysis. However, our results are stronger than Hofbauer and Sorger's in three respects. First, they assume $r > 1$ to apply the potential method. Also, they show global accessibility only for r sufficiently close to 1. Second, they do not show d-absorption (Proposition 5 of this paper), i.e., they do not exclude the possibility that a solution escapes from x^* temporarily. Third, they do not show rapid convergence. According to their proof, the rate of convergence may become slower as r gets closer to 1, whereas the time needed for exactly reaching x^* in our proof is finite and bounded from above independently of r .

²⁰See the paragraph before Proposition 3 for the definition of social stability with respect

As Hofbauer [5] points out, however, Matsui’s characterization, especially sufficiency, is invalidated by the existence of more general solutions. For example, in some non-zero-sum rock-scissors-paper game there does not exist any piecewise linear solution from the equilibrium except the constant path, but there does exist a more general solution which spirals out of the equilibrium. The solution is kinked infinitely often in a neighborhood of $t = 0$. See Hofbauer [5, Example 3.3] for details.²¹

How to choose a class of paths may affect not only the stability of an equilibrium, but also the existence of solutions. To show the existence of solutions to each dynamic, people in the literature use the class of absolutely continuous paths (Hofbauer [5] for BRD; Hofbauer and Sorger [10, 11] and Oyama [23] for PFD). Probably this is the widest class of paths in continuous-time dynamic analyses.

Thus, one may come up with two questions. (i) Are our stability results robust if we use a larger class of paths? (ii) Is our class of paths large enough for a solution to exist? About question (i), it is shown as a corollary of Hofbauer [5, Theorem 5.1] that Proposition 3 on BRD (except for the uniqueness part) still holds without the assumption of piecewise linearity: for any initial state, every solution to BRD converges to x^* in a finite time if the stage game has the anti-coordination property. For PFD we have not shown that Lemma 1 holds for more general classes of paths, but there is no known counterexample, either. About question (ii), the existence of a piecewise linear solution is guaranteed both under BRD and under PFD if the stage game has the anti-coordination property (Propositions 3 and 4).

To summarize, the assumption of piecewise linearity may or may not lose generality in the analysis of dynamic stability, but the class of piecewise linear paths is not too restrictive: It satisfies the minimum requirement that at least one solution exists under dynamics of our interest. As long as this requirement is met, whether more general paths are taken into account should be determined by how meaningful such paths are in economic or biological applications.

to BRD.

²¹Note that Hofbauer [5] also uses the term of “piecewise linearity,” but in a different meaning. According to his definition, a path is called “piecewise linear” if the set of kinks is at most countable and closed (and hence nowhere dense). The class of “piecewise linear” paths in his definition is less restrictive than our class. It is shown that his class does not lose generality as solutions to BRD [5, Theorem 2.1].

6 Conclusion

We investigated dynamic stability of anti-coordination games. If the stage game has the anti-coordination property, for any initial state, there is a unique solution to BRD, which reaches the unique symmetric equilibrium in a finite time. Under PFD, the equilibrium is stable in two senses. There exists a path from any initial state to the equilibrium, and once the path reaches the equilibrium, then the path stays there forever. For some subclasses of anti-coordination games, any solution to PFD converges to the equilibrium.

We note that our results depend explicitly or implicitly on the following assumptions: perfect foresight, a single population, exponential discounting, homogeneous action revision, the linearity of the payoff function in mixed strategies, piecewise linearity of solutions, and, above all, the anti-coordination property.²² Under these stringent assumptions, we obtained strong predictions about behavior in PFD independently of the discount rate.²³

Turning our eyes to other dynamics, we see that symmetric equilibria of anti-coordination games can be locally unstable. In Appendix, we will give an example of an anti-coordination game whose symmetric equilibrium is unstable both under the replicator dynamic and under the smoothed BRD associated with the logistic quantal response function (Fudenberg and Levine [2, Chapter 4]). Despite this instability, we point out in Appendix that other criteria of stability are satisfied in any anti-coordination game: The replicator dynamic is permanent; in the smoothed BRD, the ω -limit approaches $\{x^*\}$ as the smoothed best response tends to the exact best response.

In the study of PFD, the stability of ESS is an important open question. This is first conjectured by Hofbauer and Sorger [10, Section 6]. Although an interior ESS is known to be globally stable under BRD and many other dynamics, its stability under PFD is neither proved nor disproved.

²²By homogeneous action revision we mean that who can change his action at each moment is independent of his name and any part of the past history.

²³The anti-coordination property in Propositions 3 and 4 can be relaxed. For instance, suppose that each restricted game $G(B)$ has at least one symmetric Nash equilibrium against which every pure strategy in B is indifferent. Then there is a path from any initial state to *some* equilibrium, which is a solution both to BRD and to PFD. (This path may not be the unique solution even to BRD.) For example, this condition is satisfied in the following payoff matrix

$$\begin{pmatrix} 0 & a & b \\ b & 0 & a \\ a & b & 0 \end{pmatrix}$$

if and only if $ab \geq 0$, whereas the anti-coordination property is satisfied if and only if $a > 0$ and $b > 0$.

A Appendix

A.1 Unstable Equilibria

We give an example of an anti-coordination game whose symmetric Nash equilibrium is locally unstable both under the replicator dynamic and under a smoothed BRD.²⁴ Consider the following payoff matrix on $A = \{1, 2, 3, 4, 5\}$,

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 10 \\ 10 & 0 & 1 & 2 & 2 \\ 2 & 10 & 0 & 1 & 2 \\ 2 & 2 & 10 & 0 & 1 \\ 1 & 2 & 2 & 10 & 0 \end{pmatrix}. \quad (5)$$

Using Proposition 2 and a tedious computation, we can show that (5) satisfies the anti-coordination property. The symmetric Nash equilibrium is $x^* = (1/5, \dots, 1/5)$.

First, we consider the replicator dynamic defined by

$$\begin{aligned} \phi(0) &= x, \\ \frac{d\phi_i}{dt}(t) &= \left(\sum_j u_{ij}\phi_j(t) - \sum_{k,j} u_{kj}\phi_k(t)\phi_j(t) \right) \phi_i(t) \quad \text{for any } i \in A \text{ and } t \geq 0. \end{aligned}$$

The eigenvalues of the Jacobian at x^* are

$$\gamma_0 = -3, \quad \gamma_j = \frac{1}{5} (\omega^j + 2\omega^{2j} + 2\omega^{3j} + 10\omega^{4j}) \quad \text{for } j = 1, \dots, 4, \quad (6)$$

where $\omega = \exp(2\pi\sqrt{-1}/5)$.²⁵ (The eigenvector of γ_0 is $(1, \dots, 1)$, which is orthogonal to Δ .) Since γ_1 and γ_4 have a positive real part, x^* is unstable.

Although anti-coordination games may not have the local stability in the replicator dynamic, they satisfy a different stability property. A dynamical system is said to be *permanent* if the boundary of the state space is a repeller under the dynamic, or more precisely, if there exists $\delta > 0$ such that $\liminf_{t \rightarrow \infty} \phi_i(t) > \delta$ for each i for any interior initial state x . It is easy to check through Hofbauer and Sigmund [8, Theorem 13.6.1] that the replicator dynamic in an anti-coordination game is permanent.

Second, we consider a smoothed BRD. Let $\sigma^\lambda: \Delta \rightarrow \Delta$ be the logistic quantal response function with parameter $\lambda \geq 0$:

$$\sigma_i^\lambda(x) = \frac{\exp\left(\lambda \sum_j u_{ij}x_j\right)}{\sum_k \exp\left(\lambda \sum_j u_{kj}x_j\right)}.$$

²⁴We thank Josef Hofbauer for providing this example.

²⁵See Hofbauer and Sigmund [8, Section 14.2].

$x \in \Delta$ is a symmetric logit equilibrium with parameter λ if $x = \sigma^\lambda(x)$. As $\lambda \rightarrow \infty$, σ^λ converges (in an appropriate sense) to the best response correspondence. We define the logistic BRD with parameter λ by

$$\phi(0) = x, \quad \frac{d\phi}{dt}(t) = \sigma^\lambda(\phi(t)) - \phi(t) \quad \text{for } t \geq 0.$$

A steady state of this dynamic corresponds to a symmetric logit equilibrium. See Fudenberg and Levine [2, Chapter 4].

In game (5), $x^* = (1/5, \dots, 1/5)$ is the unique symmetric logit equilibrium for any $\lambda \geq 0$. The eigenvalues of the Jacobian at x^* are

$$\gamma'_0 = -1, \quad \gamma'_j = \lambda\gamma_j - 1 \quad \text{for } j = 1, \dots, 4,$$

where γ_j are given by (6). (The eigenvector of γ'_0 is again $(1, \dots, 1)$ and orthogonal to Δ .) Since γ_1 and γ_4 have a positive real part, γ'_1 and γ'_4 have a positive real part for sufficiently large λ . In such a case, x^* is unstable.

This result looks paradoxical at first glance: although the logistic BRD is close to the original BRD when λ is large, the stability of the equilibrium is opposite between the two dynamics. How can we resolve the paradox? Since x^* is unstable under the logistic BRD, there exists a solution which departs from a neighborhood of x^* . The solution, however, does not go completely far away from x^* . It stays near x^* , and the maximum distance from x^* tends to 0 as λ tends to ∞ . In other words, the ω -limit of the solution is not $\{x^*\}$, but is included in a small ball around x^* , which shrinks to the point x^* as $\lambda \rightarrow \infty$.

For more details about the relationship between the logistic BRD and the replicator dynamic, see Hopkins [12].

A.2 Proof of Proposition 2

Proof of Proposition 2. The only-if direction is relatively easy to show. For any $B \subsetneq A$, $G(B)$ is an anti-coordination game. Hence it has a unique symmetric Nash equilibrium $x^*(B)$ by Proposition 1. Then we have $\text{wr}(x^*(B)) = B$ since any strategy inside B is indifferent against $x^*(B)$, and worse than any strategy outside B by the anti-coordination property of G .

For the if direction, we need to show $\text{wr}(x) \subseteq \text{supp}(x)$ for any $x \in \Delta$. If x is in the interior, then this relation is trivial. If not, we fix any such x .

Then we construct $B^k \subseteq \text{supp}(x) \subsetneq A$ and $c^k > 0$ for each $k = 0, 1, 2, \dots$ by

$$\begin{aligned} B^0 &= \text{supp}(x), \\ c^k &= \min_{i \in B^k} \frac{x_i - \sum_{l=0}^{k-1} c^l x_i^*(B^l)}{x_i^*(B^k)}, \\ B^{k+1} &= B^k \setminus \arg \min_{i \in B^k} \frac{x_i - \sum_{l=0}^{k-1} c^l x_i^*(B^l)}{x_i^*(B^k)} = \text{supp} \left(x - \sum_{l=0}^k c^l x^*(B^l) \right). \end{aligned}$$

Since $B^0 \supsetneq B^1 \supsetneq B^2 \supsetneq \dots$, we stop at the m -th step when $B^{m+1} = \emptyset$. Then we obtain

$$x = \sum_{k=0}^m c^k x^*(B^k).$$

Therefore, we have

$$\text{wr}(x) = \bigcap_{k=0}^m \text{wr}(x^*(B^k)) = B^m \subseteq \text{supp}(x)$$

since $\text{wr}(x^*(B^k)) = B^k \supseteq B^m$ for each k . □

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